A GENERALIZATION OF
BANACH’S CONTRACTION PRINCIPLE

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ABSTRACT. Let $T: M \to M$ be a mapping of a metric space $(M, d)$ into itself. A mapping $T$ will be called a quasi-contraction iff $d(Tx, Ty) \le q \cdot \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}$ for some $q < 1$ and all $x, y \in M$. In the present paper the mappings of this kind are investigated. The results presented here show that the condition of quasi-contractivity implies all conclusions of Banach’s contraction principle. Multi-valued quasi-contractions are also discussed.

1. Introduction. The well-known Banach’s contraction mapping principle states that if $T: M \to M$ is a contraction on $M$ (i.e. $d(Tx, Ty) \le q \cdot d(x, y)$ for some $q < 1$ and all $x, y \in M$) and $M$ is complete, then

$(1^o)$ $T$ has a unique fixed point $u$ in $M$,
$(2^o)$ $\lim\limits_{n \to \infty} T^n x = u$, and
$(3^o)$ $d(T^n x, u) \le q^n (1 - q)^{-1} d(x, Tx)$ for every $x \in M$.

A number of generalizations of this result have appeared [1], [2], [3], [7], [8], [9], [12]. In [2] we considered generalized contractions, defined as follows.

A mapping $T: M \to M$ is said to be a generalized contraction iff for every $x, y \in M$ there exist nonnegative numbers $q, r, s$ and $t$, which may depend on both $x$ and $y$, such that $\sup\{q + r + s + 2t: x, y \in M\} < 1$ and

$$d(Tx, Ty) \le q \cdot d(x, y) + r \cdot d(x, Tx)$$

$$+ s \cdot d(y, Ty) + t \cdot [d(x, Ty) + d(y, Tx)].$$

S. Nadler [10] has extended Banach’s contraction principle to multi-valued contractions. Many extensions of Nadler’s result have been derived in recent years [4], [6], [11], [13]. In [4] we proved some fixed-point theorems for a class of multi-valued generalized contractions—the maps which include the single-valued generalized contractions.

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The purpose of this paper is to extend some results concerning generalized contractions of [2] and [4] to quasi-contractions. In §2 fixed-point theorems for single-valued quasi-contractions are proved and an example is given to show that the results established here are indeed extensions. In §3 it is shown that for multi-valued quasi-contractions a similar result is valid.

2. Quasi-contractions. Let $T$ be a mapping of a metric space $M$ into itself. For $A \subset M$ let $\delta(A) = \sup\{d(a, b); a, b \in A\}$ and for each $x \in M$, let

$$O(x, n) = \{x, Tx, \ldots, T^n x\}, \quad n = 1, 2, \ldots,$$

$$O(x, \infty) = \{x, Tx, \ldots\}.$$

A space $M$ is said to be $T$-orbitally complete iff every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in M$ converges in $M$ (cf. [5]).

Definition 1. A mapping $T : M \to M$ of a metric space $M$ into itself is said to be a quasi-contraction iff there exists a number $q$, $0 \leq q < 1$, such that

$$(B) \ d(Tx, Ty) \leq q \cdot \max\{d(x, y); d(x, Tx); d(y, Ty)\}$$

holds for every $x, y \in M$.

It is clear that condition (A) implies (B). The following example shows that a quasi-contraction need not be a generalized contraction.

Example. Let

$$M_1 = \{m/n; m = 0, 1, 3, 9, \ldots; n = 1, 4, \ldots, 3k + 1, \ldots\},$$

$$M_2 = \{m/n; m = 1, 3, 9, 27, \ldots; n = 2, 5, \ldots, 3k + 2, \ldots\},$$

and let $M = M_1 \cup M_2$ with the usual metric. Define $T : M \to M$ by

$$Tx = 3x/5, \quad \text{for } x \in M_1,$$

$$Tx = x/8, \quad \text{for } x \in M_2.$$

The mapping $T$ is a quasi-contraction with $q = 3/5$. Indeed, if both $x$ and $y$ are in $M_1$ or in $M_2$, then $d(Tx, Ty) \leq (3/5)d(x, y)$. Now let $x$ be, for example, in $M_1$ and $y$ in $M_2$. Then

$$x > \frac{5}{24}y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5}\left(x - \frac{5}{24}y\right) \leq \frac{3}{5}\left(x - \frac{1}{8}y\right) = \frac{3}{5}d(x, Ty);$$

$$x < \frac{5}{24}y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5}\left(x - \frac{5}{24}y\right) \leq \frac{3}{5}(y - x) = \frac{3}{5}d(x, y).$$

Therefore, $T$ on $M$ satisfies the condition.
and hence (B).

To show that $T$ is not a generalized contraction on $M$, let $x = 1$ and $y = \frac{1}{2}$. Then we have

$$q \cdot d(x, y) + r \cdot d(x, Tx) + s \cdot d(y, Ty) + t \cdot [d(x, Ty) + d(y, Tx)]$$

$$= q \cdot \frac{1}{2} + r \cdot \frac{2}{5} + s \cdot \frac{7}{16} + t \cdot \frac{83}{80}$$

$$< (q + r + s + 2t) \cdot \frac{83}{160} < \frac{83}{80} < \frac{43}{40} = d(Tx, Ty),$$

as $q + r + s + 2t < 1$, and we see that condition (A) is not satisfied.

Before stating the fixed-point theorem for quasi-contractions we shall prove two lemmas on these mappings. The first of these lemmas is fundamental.

**Lemma 1.** Let $T$ be a quasi-contraction on $M$ and let $n$ be any positive integer. Then for each $x \in M$ and all positive integers $i$ and $j$, $i, j \in \{1, 2, \cdots, n\}$ implies $d(T^i x, T^j x) \leq q \cdot \delta[0(x, n)]$.

**Proof.** Let $x \in M$ be arbitrary, let $n$ be any positive integer and let $i$ and $j$ satisfy the condition of Lemma 1. Then $T^i x, T^j x, T^i x, T^j x \in \delta[0(x, n)]$ (where it is understood that $T^0 x = x$) and since $T$ is a quasi-contraction, we have

$$d(T^i x, T^j x) = d(TT^i x, TT^j x)$$

$$\leq q \cdot \max \{d(T^i x, T^j x); d(T^i x, T^i x); d(T^i x, T^j x);$$

$$d(T^j x, T^i x); d(T^i x, T^j x); d(T^j x, T^i x)\}$$

$$\leq q \cdot \delta[0(x, n)],$$

which proves the lemma.

**Remark.** From this lemma it follows that if $T$ is a quasi-contraction and $x \in M$, then for every positive integer $n$ there exists a positive integer $k \leq n$, such that $d(x, T^k x) = \delta[0(x, n)]$.

**Lemma 2.** If $T$ is a quasi-contraction on $M$, then

$$\delta[0(x, \infty)] \leq (1/(1 - q))d(x, Tx)$$

holds for all $x \in M$.

**Proof.** Let $x \in M$ be arbitrary. Since $\delta[0(x, 1)] \leq \delta[0(x, 2)] \leq \cdots$, we
see that \( \delta[O(x, \infty)] = \sup\{\delta[O(x, n)]; n \in \mathbb{N}\} \). The lemma will follow if we show that \( \delta[O(x, n)] \leq (1/(1 - q))d(x, Tx) \) for all \( n \in \mathbb{N} \).

Let \( n \) be any positive integer. From the remark to the previous lemma, there exists \( T^kx \in O(x, n) \) (\( 1 \leq k \leq n \)) such that \( d(x, T^kx) = \delta[O(x, n)] \).

Applying a triangle inequality and Lemma 1, we get
\[
d(x, T^kx) \leq d(x, Tx) + d(Tx, T^kx) \leq d(x, Tx) + q \cdot \delta[O(x, n)]
\]
\[
= d(x, Tx) + q \cdot d(x, T^kx).
\]

Therefore, \( \delta[O(x, n)] = d(x, T^kx) \leq (1/(1 - q))d(x, Tx) \). Since \( n \) was arbitrary, the proof is completed.

Now we can state our main result.

**Theorem 1.** Let \( T \) be a quasi-contraction on a metric space \( M \) and let \( M \) be \( T \)-orbitally complete. Then

(a) \( T \) has a unique fixed point \( u \) in \( M \),

(b) \( \lim_{n \to \infty} T^n x = u \), and

(c) \( d(T^n x, u) \leq (q^n/(1 - q))d(x, Tx) \) for every \( x \in M \).

**Proof.** Let \( x \) be an arbitrary point of \( M \). We shall show that the sequence of iterates \( \{T^n x\} \) is a Cauchy sequence.

Let \( n \) and \( m (n < m) \) be any positive integers. Since \( T \) is a quasi-contraction, it follows from Lemma 1 that
\[
d(T^n x, T^m x) = d(TT^{n-1} x, T^{m-n+1}T^{n-1} x) \leq q \cdot \delta[O(T^{n-1} x, m - n + 1)].
\]

According to the remark to Lemma 1, there exists an integer \( k_1, 1 \leq k_1 \leq m - n + 1 \), such that
\[
\delta[O(T^{n-1} x, m - n + 1)] = d(T^{n-1} x, T^{k_1}T^{n-1} x).
\]

Again, by Lemma 1, we have
\[
d(T^{n-1} x, T^{k_1}T^{n-1} x) = d(TT^{n-2} x, T^{k_1+1}T^{n-2} x)
\]
\[
\leq q \cdot \delta[O(T^{n-2} x, k_1 + 1)]
\]
\[
\leq q \cdot \delta[O(T^{n-2} x, m - n + 2)].
\]

Therefore, we have the following system of inequalities.
\[
d(T^n x, T^m x) \leq q \cdot \delta[O(T^{n-1} x, m - n + 1)] \leq q^2 \cdot \delta[O(T^{n-2} x, m - n + 2)].
\]

Proceeding in this manner, we obtain
Then it follows from Lemma 2 that

\[ d(T^n x, T^m x) \leq q \cdot \delta[O(T^n x, T^m x)] \leq \cdots \leq q^n \cdot \delta[O(x, m)]. \]

Since \( \lim_n q^n = 0 \), \( \{T^n x\} \) is a Cauchy sequence.

Again, \( M \) being \( T \)-orbitally complete, \( \{T^n x\} \) has a limit \( u \) in \( M \). To prove that \( Tu = u \), let us consider the following inequalities.

\[ d(u, Tu) \leq d(u, T^{n+1} x) + d( TT^n x, Tu) \]

\[ \leq d(u, T^{n+1} x) + q \cdot \max\{d(T^n x, u), d(T^n x, T^{n+1} x); d(u, Tu); d(T^n x, Tu); d(T^{n+1} x, u)\} \]

\[ \leq d(u, T^{n+1} x) + q \cdot [d(T^n x, T^{n+1} x) + d(T^n x, u) \]

\[ + d(u, Tu) + d(T^{n+1} x, u)]. \]

Hence

\[ d(u, Tu) \leq \frac{1}{1 - q} [(1 + q)d(u, T^{n+1} x) + q \cdot d(u, T^n x) + q \cdot d(T^n x, T^{n+1} x)]. \]

Since \( \lim_n T^n x = u \), this shows that \( d(u, Tu) = 0 \), i.e. \( u \) is a fixed point under \( T \). The uniqueness follows from the quasi-contractivity of \( T \). So we have proved (a) and (b), as \( x \) was arbitrary. Letting \( m \) tend to infinity in (1), we obtain the inequality (c).

This completes the proof of the theorem.

The next result readily follows from the above theorem.

**Theorem 2.** Let \( T \) be a mapping of a metric space \( M \) into itself and let \( M \) be \( T \)-orbitally complete. If there exists a positive integer \( k \) such that the iteration \( T^k \) is a quasi-contraction, then

(a') \( T \) has a unique fixed point \( u \) in \( M \),

(b') \( \lim_n T^n x = u \), and

(c') \( d(T^n x, u) \leq q^m a(x)/(1 - q) \) for every \( x \in M \),

where \( a(x) = \max\{d(T^i x, T^{i+k} x): i = 0, 1, \cdots, k - 1\} \) and \( m = E(n/k) \) is the greatest integer not exceeding \( n/k \).

Proof. Since \( T^k \) has a unique fixed point \( u \) and \( T^k(Tu) = T(T^k u) = Tu \), it follows that \( Tu = u \). Its uniqueness is obvious. To show (c'), let \( n \) be any integer. Then \( n = m \cdot k + j \), \( 0 \leq j < k \), \( m \geq 0 \), and for every \( x \in M \), \( T^n x = (T^k)^m T^j x \). Since \( T^k \) is a quasi-contraction, it follows from part (c) of Theorem 1.1 that
\[ d(T^n x, u) \leq \frac{q^m}{1 - q} d(T^i x, T^k T^i x) \]
\[ \leq \frac{q^m}{1 - q} \max\{d(T^i x, T^k T^i x): i = 0, 1, \ldots, k - 1\}, \]

which proves \((c')\), and hence \((b')\). This completes the proof of the theorem.

Note that Theorem 2.5 (Theorem 2.6) of [2] is a special case of Theorem 1 (Theorem 2). The example following Definition 1 shows that Theorem 1 is more general than Theorem 2.5 of [2]. In that example \(M\) is \(T\)-orbitally complete and \(o\) is a fixed point under \(T\).

3. Multi-valued quasi-contractions. We shall now recall some terminologies. Let \((M, d)\) be a metric space and let \(A, B\) be any subsets of \(M\). We denote \(D(A, B) = \inf\{d(a, b): a \in A, b \in B\}\), \(\rho(A, B) = \sup\{d(a, b): a \in A, b \in B\}\), \(\mathcal{BN}(M) = \{A: \emptyset \neq A \subset M\}\) and \(\delta(A) < +\infty\). Let \(F: M \rightarrow M\) be a point to set correspondence and let \(x_0 \in M\). An orbit of \(F\) at \(x_0\) is a sequence \(\{x_n: x_n \in Fx_{n-1}, n = 1, 2, \ldots\}\). A space \(M\) is said to be \(F\)-orbitally complete iff every Cauchy sequence which is a subsequence of an orbit of \(F\) at \(x\) for some \(x \in M\), converges in \(M\). Among the results established in [4] was the following: if \(F: M \rightarrow \mathcal{BN}(M)\) satisfies

\[(C) \rho(Fx, Fy) \leq q \cdot \max\{d(x, y); \rho(x, Fx); \rho(y, Fy); \frac{1}{2}[D(x, Fy) + D(y, Fx)]\}\]

for some \(q < 1\) and if \(M\) is \(F\)-orbitally complete, then \(F\) has a unique fixed point \(u\) with \(Fu = \{u\}\) and for each \(x_0 \in M\) there exists an orbit \(\{x_n\}\) of \(F\) at \(x_0\) such that \(\lim x_n = u\). The following is an extension of the above statement.

**Theorem 3.** Let \(F: M \rightarrow \mathcal{BN}(M)\) be a multi-valued mapping on a metric space \(M\) and let \(M\) be \(F\)-orbitally complete. If \(F\) satisfies

\[(D) \rho(Fx, Fy) \leq q \cdot \max\{d(x, y); \rho(x, Fx); \rho(y, Fy); D(x, Fy); D(y, Fx)\}\]

for some \(q < 1\) and all \(x, y \in M\), then

(i) \(F\) has a unique fixed point \(u\) in \(M\) and \(Fu = \{u\}\),

(ii) for each \(x_0 \in M\) there exists an orbit \(\{x_n\}\) of \(F\) at \(x_0\) such that

\[\lim x_n = u,\]

(iii) \(d(x_n, u) \leq (q^{1-a}a^n/(1 - q^{1-a}))d(x_0, x_1)\),

where \(a < 1\) is any fixed positive number.

**Proof.** Let \(a \in (0, 1)\) be any number. Define a single-valued mapping \(T: M \rightarrow M\) as follows: for each \(x \in M\) let \(Tx\) be a point of \(Fx\), which satisfies

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$d(x, Tx) \geq q^a \cdot \rho(x, Fx)$. A mapping $T$ is then a quasi-contraction with $q_1 = q^{1-a}$. Indeed, for every $x, y \in M$ we have

$$d(Tx, Ty) \leq \rho(Fx, Fy)$$

$$\leq q \cdot q^{-a} \max\{q^a d(x, y); q^a \rho(x, Fx); q^a \rho(y, Fy); q^a D(x, Fy); q^a D(y, Fx)\}$$

$$\leq q^{1-a} \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\},$$

which means that $T$ is a quasi-contraction. Clearly $u = Tu$ implies $u \in Fu$. Since $F$ satisfies (D), $u \in Fu$ implies $\rho(Fu, Fu) \leq q \cdot \rho(u, Fu)$. This may happen only if $Fu = \{u\}$. Therefore, $u \in M$ is a fixed point of $T$ iff $u$ is a fixed point of $F$. Since for each $x \in M$ the sequence $\{T^n x\}$ is an orbit of $F$ at $x$, the statements of Theorem 3 follow from Theorem 1.

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