A GENERALIZATION OF BANACH'S CONTRACTION PRINCIPLE

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ABSTRACT. Let \( T: M \to M \) be a mapping of a metric space \((M, d)\) into itself. A mapping \( T \) will be called a quasi-contraction iff \( d(Tx, Ty) \leq q \cdot \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\} \) for some \( q < 1 \) and all \( x, y \in M \). In the present paper the mappings of this kind are investigated. The results presented here show that the condition of quasi-contraction implies all conclusions of Banach's contraction principle. Multi-valued quasi-contractions are also discussed.

1. Introduction. The well-known Banach's contraction mapping principle states that if \( T: M \to M \) is a contraction on \( M \) (i.e. \( d(Tx, Ty) \leq q \cdot d(x, y) \) for some \( q < 1 \) and all \( x, y \in M \)) and \( M \) is complete, then

- (1°) \( T \) has a unique fixed point \( u \) in \( M \),
- (2°) \( \lim_{n \to \infty} T^n x = u \), and
- (3°) \( d(T^n x, u) \leq q^n (1 - q)^{-1} d(x, Tx) \) for every \( x \in M \).

A number of generalizations of this result have appeared [1], [2], [3], [7], [8], [9], [12]. In [2] we considered generalized contractions, defined as follows.

A mapping \( T: M \to M \) is said to be a generalized contraction iff for every \( x, y \in M \) there exist nonnegative numbers \( q, r, s \) and \( t \), which may depend on both \( x \) and \( y \), such that \( \sup\{q + r + s + 2t: x, y \in M\} < 1 \) and

\[
d(Tx, Ty) \leq q \cdot d(x, y) + r \cdot d(x, Tx) + s \cdot d(y, Ty) + t \cdot [d(x, Ty) + d(y, Tx)].
\]

S. Nadler [10] has extended Banach's contraction principle to multi-valued contractions. Many extensions of Nadler's result have been derived in recent years [4], [6], [11], [13]. In [4] we proved some fixed-point theorems for a class of multi-valued generalized contractions—the maps which include the single-valued generalized contractions.

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The purpose of this paper is to extend some results concerning generalized contractions of [2] and [4] to quasi-contractions. In §2 fixed-point theorems for single-valued quasi-contractions are proved and an example is given to show that the results established here are indeed extensions. In §3 it is shown that for multi-valued quasi-contractions a similar result is valid.

2. Quasi-contractions. Let $T$ be a mapping of a metric space $M$ into itself. For $A \subseteq M$ let $\delta(A) = \sup \{d(a, b) : a, b \in A\}$ and for each $x \in M$, let

\[ O(x, n) = \{x, Tx, \ldots, T^n x\}, \quad n = 1, 2, \ldots, \]
\[ O(x, \infty) = \{x, Tx, \ldots\}. \]

A space $M$ is said to be $T$-orbitally complete iff every Cauchy sequence which is contained in $O(x, \infty)$ for some $x \in M$ converges in $M$ (cf. [5]).

Definition 1. A mapping $T : M \to M$ of a metric space $M$ into itself is said to be a quasi-contraction iff there exists a number $q, 0 \leq q < 1$, such that

(B) $d(Tx, Ty) \leq q \cdot \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}$ holds for every $x, y \in M$.

It is clear that condition (A) implies (B). The following example shows that a quasi-contraction need not be a generalized contraction.

Example. Let

\[ M_1 = \{m/n : m = 0, 1, 3, 9, \ldots ; n = 1, 4, \ldots, 3k + 1, \ldots\}, \]
\[ M_2 = \{m/n : m = 1, 3, 9, 27, \ldots ; n = 2, 5, \ldots, 3k + 2, \ldots\}, \]

and let $M = M_1 \cup M_2$ with the usual metric. Define $T : M \to M$ by

\[ Tx = 3x/5, \quad \text{for } x \in M_1, \]
\[ = x/8, \quad \text{for } x \in M_2. \]

The mapping $T$ is a quasi-contraction with $q = 3/5$. Indeed, if both $x$ and $y$ are in $M_1$ or in $M_2$, then $d(Tx, Ty) \leq (3/5)d(x, y)$. Now let $x$ be, for example, in $M_1$ and $y$ in $M_2$. Then

\[ x > \frac{5}{24} y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5} \left( x - \frac{5}{24} y \right) \leq \frac{3}{5} \left( x - \frac{1}{8} y \right) = \frac{3}{5} d(x, Ty); \]
\[ x < \frac{5}{24} y \quad \text{implies} \quad d(Tx, Ty) = \frac{3}{5} \left( \frac{5}{24} y - x \right) \leq \frac{3}{5} (y - x) = \frac{3}{5} d(x, y). \]

Therefore, $T$ on $M$ satisfies the condition.
\[ d(Tx, Ty) \leq (3/5) \max \{ d(x, y); d(x, Ty); d(y, Tx) \} \]

and hence (B).

To show that \( T \) is not a generalized contraction on \( M \), let \( x = 1 \) and \( y = \frac{1}{2} \). Then we have

\[
q \cdot d(x, y) + r \cdot d(x, Tx) + s \cdot d(y, Ty) + t \cdot (d(x, Ty) + d(y, Tx))
\]

\[
= q \cdot \frac{1}{2} + r \cdot \frac{2}{5} + s \cdot \frac{7}{16} + t \cdot \frac{83}{80}
\]

\[
< (q + r + s + 2t) \cdot \frac{83}{160} < \frac{83}{160} < \frac{43}{80} = d(Tx, Ty),
\]

as \( q + r + s + 2t < 1 \), and we see that condition (A) is not satisfied.

Before stating the fixed-point theorem for quasi-contractions we shall prove two lemmas on these mappings. The first of these lemmas is fundamental.

**Lemma 1.** Let \( T \) be a quasi-contraction on \( M \) and let \( n \) be any positive integer. Then for each \( x \in M \) and all positive integers \( i \) and \( j \), \( i, j \in \{1, 2, \ldots, n\} \) implies \( d(T^ix, T^jx) \leq q \cdot \delta[O(x, n)] \).

**Proof.** Let \( x \in M \) be arbitrary, let \( n \) be any positive integer and let \( i \) and \( j \) satisfy the condition of Lemma 1. Then \( T^{i-1}x, T^ix, T^{j-1}x, T^jx \in O(x, n) \) (where it is understood that \( T^0x = x \)) and since \( T \) is a quasi-contraction, we have

\[
d(T^ix, T^jx) = d(TT^{i-1}x, TT^{j-1}x)
\]

\[
\leq q \cdot \max \{ d(T^{i-1}x, T^{j-1}x); d(T^{i-1}x, T^ix); d(T^{j-1}x, T^jx); d(T^ix, T^{j-1}x) \}
\]

\[
\leq q \cdot \delta[O(x, n)],
\]

which proves the lemma.

**Remark.** From this lemma it follows that if \( T \) is a quasi-contraction and \( x \in M \), then for every positive integer \( n \) there exists a positive integer \( k \leq n \), such that \( d(x, T^kx) = \delta[O(x, n)] \).

**Lemma 2.** If \( T \) is a quasi-contraction on \( M \), then

\[
\delta[O(x, \infty)] \leq (1/(1 - q))d(x, Tx)
\]

holds for all \( x \in M \).

**Proof.** Let \( x \in M \) be arbitrary. Since \( \delta[O(x, 1)] \leq \delta[O(x, 2)] \leq \cdots \), we
see that \( \delta[O(x, \infty)] = \sup \{\delta[O(x, n)] : n \in \mathbb{N}\} \). The lemma will follow if we show that \( \delta[O(x, n)] \leq (1/(1 - q))d(x, Tx) \) for all \( n \in \mathbb{N} \).

Let \( n \) be any positive integer. From the remark to the previous lemma, there exists \( T^kx \in O(x, n) (1 \leq k \leq n) \) such that \( d(x, T^kx) = \delta[O(x, n)] \).

Applying a triangle inequality and Lemma 1, we get

\[
d(x, T^kx) \leq d(x, Tx) + d(Tx, T^kx) \leq d(x, Tx) + q \cdot \delta[O(x, n)]
\]

\[
= d(x, Tx) + q \cdot d(x, T^kx).
\]

Therefore, \( \delta[O(x, n)] = d(x, T^kx) \leq (1/(1 - q))d(x, Tx) \). Since \( n \) was arbitrary, the proof is completed.

Now we can state our main result.

**Theorem 1.** Let \( T \) be a quasi-contraction on a metric space \( M \) and let \( M \) be \( T \)-orbitally complete. Then

(a) \( T \) has a unique fixed point \( u \) in \( M \),

(b) \( \lim_{n \to \infty} T^n x = u \), and

(c) \( d(T^n x, u) \leq (q^n/(1 - q))d(x, Tx) \) for every \( x \in M \).

**Proof.** Let \( x \) be an arbitrary point of \( M \). We shall show that the sequence of iterates \( \{T^n x\} \) is a Cauchy sequence.

Let \( n \) and \( m \) \((n < m)\) be any positive integers. Since \( T \) is a quasi-contraction, it follows from Lemma 1 that

\[
d(T^n x, T^m x) = d(TT^{n-1} x, T^{m-n+1} T^{n-1} x) \leq q \cdot \delta[O(T^{n-1} x, m - n + 1)].
\]

According to the remark to Lemma 1, there exists an integer \( k_1, 1 \leq k_1 \leq m - n + 1 \), such that

\[
\delta[O(T^{n-1} x, m - n + 1)] = d(T^{n-1} x, T^{k_1} T^{n-1} x).
\]

Again, by Lemma 1, we have

\[
d(T^{n-1} x, T^{k_1} T^{n-1} x) = d(TT^{n-2} x, T^{k_1+1} T^{n-2} x)
\]

\[
\leq q \cdot \delta[O(T^{n-2} x, k_1 + 1)]
\]

\[
\leq q \cdot \delta[O(T^{n-2} x, m - n + 2)].
\]

Therefore, we have the following system of inequalities.

\[
d(T^n x, T^m x) \leq q \cdot \delta[O(T^{n-1} x, m - n + 1)] \leq q^2 \cdot \delta[O(T^{n-2} x, m - n + 2)].
\]

Proceeding in this manner, we obtain
Then it follows from Lemma 2 that

\[
d(T^n x, T^m x) \leq (q^n/(1 - q))d(x, Tx).
\]

Since \( \lim_n q^n = 0 \), \( \{T^n x\} \) is a Cauchy sequence.

Again, \( M \) being \( T \)-orbitally complete, \( \{T^n x\} \) has a limit \( u \) in \( M \). To prove that \( Tu = u \), let us consider the following inequalities.

\[
d(u, Tu) \leq d(u, T^{n+1} x) + d(T^n x, Tu)
\]

\[
\leq d(u, T^{n+1} x) + q \cdot \max\{d(T^n x, u), d(T^n x, T^{n+1} x)\}
\]

\[
\leq d(u, T^{n+1} x) + q \cdot [d(T^n x, T^{n+1} x) + d(T^n x, u)]
\]

Hence

\[
d(u, Tu) \leq \frac{1}{1 - q} [(1 + q)d(u, T^{n+1} x) + q \cdot d(u, T^n x) + q \cdot d(T^n x, T^{n+1} x)].
\]

Since \( \lim_n T^n x = u \), this shows that \( d(u, Tu) = 0 \), i.e., \( u \) is a fixed point under \( T \). The uniqueness follows from the quasi-contractivity of \( T \). So we have proved (a) and (b), as \( x \) was arbitrary. Letting \( m \) tend to infinity in (1), we obtain the inequality (c).

This completes the proof of the theorem.

The next result readily follows from the above theorem.

**Theorem 2.** Let \( T \) be a mapping of a metric space \( M \) into itself and let \( M \) be \( T \)-orbitally complete. If there exists a positive integer \( k \) such that the iteration \( T^k \) is a quasi-contraction, then

(a') \( T \) has a unique fixed point \( u \) in \( M \),

(b') \( \lim_n T^n x = u \), and

(c') \( d(T^n x, u) \leq q^m a(x)/(1 - q) \) for every \( x \in M \),

where \( a(x) = \max\{d(T^i x, T^{i+k} x) : i = 0, 1, \ldots, k - 1\} \) and \( m = E(n/k) \) is the greatest integer not exceeding \( n/k \).

**Proof.** Since \( T^k \) has a unique fixed point \( u \) and \( T^k(Tu) = T(T^k u) = Tu \), it follows that \( Tu = u \). Its uniqueness is obvious. To show (c'), let \( n \) be any integer. Then \( n = m \cdot k + j, 0 \leq j < k, m \geq 0 \), and for every \( x \in M \), \( T^n x = (T^k)^m T^j x \). Since \( T^k \) is a quasi-contraction, it follows from part (c) of Theorem 1 that...
\[ d(T^n x, u) \leq \frac{q^m}{1 - q} d(T^i x, T^k T^i x) \]
\[ \leq \frac{q^m}{1 - q} \max \{d(T^i x, T^k T^i x): i = 0, 1, \ldots, k - 1\}, \]

which proves (c'), and hence (b'). This completes the proof of the theorem.

Note that Theorem 2.5 (Theorem 2.6) of [2] is a special case of Theorem 1 (Theorem 2). The example following Definition 1 shows that Theorem 1 is more general than Theorem 2.5 of [2]. In that example M is T-orbitally complete and \( \theta \) is a fixed point under \( T \).

3. Multi-valued quasi-contractions. We shall now recall some terminologies. Let \( (M, d) \) be a metric space and let \( A, B \) be any subsets of \( M \). We denote \( D(A, B) = \inf \{d(a, b): a \in A, b \in B\} \), \( \rho(A, B) = \sup \{d(a, b): a \in A, b \in B\} \), \( BN(M) = \{A: \emptyset \neq A \subset M \text{ and } \delta(A) < +\infty\} \). Let \( F: M \to M \) be a point to set correspondence and let \( x_0 \in M \). An orbit of \( F \) at \( x_0 \) is a sequence \( \{x_n: x_n \in F x_{n-1}, n = 1, 2, \ldots\} \). A space \( M \) is said to be \( F \)-orbitally complete iff every Cauchy sequence which is a subsequence of an orbit of \( F \) at \( x \) for some \( x \in M \), converges in \( M \). Among the results established in [4] was the following: if \( F: M \to BN(M) \) satisfies

\[ \rho(Fx, Fy) \leq q \cdot \max \{d(x, y); \rho(x, Fx); \rho(y, Fy); \frac{1}{2}[D(x, Fy) + D(y, Fx)]\} \]

for some \( q < 1 \) and if \( M \) is \( F \)-orbitally complete, then \( F \) has a unique fixed point \( u \) with \( Fu = \{u\} \) and for each \( x_0 \in M \) there exists an orbit \( \{x_n\} \) of \( F \) at \( x_0 \) such that \( \lim x_n = u \). The following is an extension of the above statement.

**Theorem 3.** Let \( F: M \to BN(M) \) be a multi-valued mapping on a metric space \( M \) and let \( M \) be \( F \)-orbitally complete. If \( F \) satisfies

\[ \rho(Fx, Fy) \leq q \cdot \max \{d(x, y); \rho(x, Fx); \rho(y, Fy); D(x, Fy); D(y, Fx)\} \]

for some \( q < 1 \) and all \( x, y \in M \), then

(i) \( F \) has a unique fixed point \( u \) in \( M \) and \( Fu = \{u\} \),

(ii) for each \( x_0 \in M \) there exists an orbit \( \{x_n\} \) of \( F \) at \( x_0 \) such that \( \lim x_n = u \), and

(iii) \( d(x_n, u) \leq \left((q^{1-a}n/(1 - q^{1-a}))d(x_0, x_1), \right. \)

where \( a < 1 \) is any fixed positive number.

**Proof.** Let \( a \in (0, 1) \) be any number. Define a single-valued mapping \( T: M \to M \) as follows: for each \( x \in M \) let \( Tx \) be a point of \( Fx \), which satisfies
d(x, Tx) ≥ q^a · ρ(x, Fx). A mapping T is then a quasi-contraction with q_1 = q^{1-a}. Indeed, for every x, y ∈ M we have
\[ d(Tx, Ty) ≤ p(Fx, Fy) \]
\[ ≤ q · q^{-a} \max\{q^a d(x, y); q^a ρ(x, Fx); q^a ρ(y, Fy); q^a D(x, Fy); q^a D(y, Fx)\} \]
\[ ≤ q^{1-a} \max\{d(x, y); d(x, Tx); d(y, Ty); d(x, Ty); d(y, Tx)\}, \]
which means that T is a quasi-contraction. Clearly u = Tu implies u ∈ Fu. Since F satisfies (D), u ∈ Fu implies p(Fu, Fu) ≤ q · p(u, Fu). This may happen only if Fu = {u}. Therefore, u ∈ M is a fixed point of T iff u is a fixed point of F. Since for each x ∈ M the sequence \{T^n x\} is an orbit of F at x, the statements of Theorem 3 follow from Theorem 1.

REFERENCES