SUMMABILITY METHODS FAIL FOR THE 2ⁿTH PARTIAL SUMS OF FOURIER SERIES

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ABSTRACT. Although the Fourier series of a continuous function need not converge everywhere, it was an important discovery of Fejér that this series must be Cesàro summable. Indeed, it is a frequent occurrence that convergence may be restored to an expansion by use of an appropriate summability method. What we show in this note is that the very opposite phenomenon can occur. Namely, that if one considers only the 2ⁿth partial sums of the Fourier series, there is no summability method whatever which produces convergence for all continuous functions.

We need only consider regular matrix summability methods since a "sequence-to-function" method may be viewed through a sequence of the continuous variable (divergence through this subsequence surely implies divergence through the continuous variable).

Theorem. Let N denote the summability method which sends xₙ into n ²⁻ⁿ. If M is any regular summability method, then M • N is not effective for Fourier series.

Proof. Let M have the matrix mᵢ, so that j=1 |mᵢ| ≤ c for all i, lim i→∞ j=1 mᵢ = 1, lim i→∞ mᵢ = 0 for all j. Our assertion is equivalent to the fact that j=1 mᵢ sin 2⁻ⁿ x |dx/x| is unbounded as i → ∞ and we show that, indeed, it goes to ∞. Now our conditions on mᵢ surely imply that j=n ∑ mᵢ = oo. For we have

i=1 j=n ∑ mᵢ ≥ ∑ j>n mᵢ ≥ (∑ j=n mᵢ)² (∑ j=n 1/j²)⁻¹,

by the Schwarz inequality, and this is ≥ n(∑ j=n mᵢ)². Thus, for any fixed n,
Our theorem therefore follows from the Lemma. \( \int_0^{\pi} |\sum a_j \sin 2^j x| \, (dx/x) \geq (1/5)(\sum j^2 a_j^2)^{1/2} - \pi \sum |a_j|. \)

For proof, split the given integral into \( \sum_{k=1}^{\infty} l_k \) with

\[
l_k = \int_{\pi \cdot 2^{-k}}^{2\pi \cdot 2^{-k}} |\sum a_j \sin 2^j x| \, \frac{dx}{x}.
\]

We have then

\[
l_k = \int_{\pi}^{2\pi} \left| \sum_{j \geq k} a_j \sin 2^{j-k} x \right| \, \frac{dx}{x} \geq J_k - \delta_k
\]

where

\[
J_k = \int_{\pi}^{2\pi} \left| \sum_{j \geq k} a_j \sin 2^{j-k} x \right| \, \frac{dx}{x}, \quad \delta_k = \int_{\pi}^{2\pi} \left| \sum_{j < k} a_j \sin 2^{j-k} x \right| \, \frac{dx}{x}.
\]

Regarding \( \delta_k \) we note that

\[
\delta_k \leq \int_{\pi}^{2\pi} \sum_{j < k} |a_j| \cdot 2^j \, \sin x \cdot \frac{dx}{x} = \pi \sum_{j < k} |a_j| \cdot 2^j
\]

so that

\[
\sum_{k=1}^{\infty} \delta_k \leq \sum_{k=1}^{\infty} \sum_{j < k} |a_j| \cdot 2^j = \pi \sum_{j} |a_j| \sum_{k > j} 2^j = \pi \sum_{j} |a_j| \cdot 1.
\]

As for the \( J_k \) we have

\[
2J_k \geq 2 \int_{\pi}^{2\pi} \left| \sum_{j \geq k} a_j \sin 2^{j-k} t \right| \, dt \cdot \frac{1}{2\pi}
\]

\[
= \frac{1}{\pi} \int_0^{\pi} |a_k \sin t + a_{k+1} \sin 2t + a_{k+2} \sin 4t + \cdots| \, dt,
\]

and this expression is the \( L^1 \) norm of a trigonometric gap series. For such series the \( L^1 \) and \( L^2 \) norms are equivalent [1]. Indeed in the present case we actually obtain \( J_k \geq (1/5)(\sum j^2 a_j^2)^{1/2} \).
Our Lemma now follows from the following simple elementary inequality

\[ \sum_{k=1}^{\infty} \left( \sum_{j \geq k} a_j^2 \right)^{\frac{1}{2}} \geq \left( \sum_{j=1}^{\infty} j^2 a_j^2 \right)^{\frac{1}{2}}. \]

Indeed, calling \( R_k = \left( \sum_{j \geq k} a_j^2 \right)^{\frac{1}{2}} \), we have, by the monotonicity of \( R_k \)

\[ \left( \sum_{k=1}^{\infty} R_k \right)^2 = \sum_{k=1}^{\infty} R_k \left( 2 \sum_{i<k} R_i + R_k \right) \geq \sum_{k=1}^{\infty} R_k^2 (2k-1)R_k + R_k. \]

In turn this is equal to

\[ \sum_{k=1}^{\infty} (2k-1)R_k^2 = \sum_{k=1}^{\infty} (2k-1) \sum_{j \geq k} a_j^2 = \sum_{j} a_j^2 \sum_{k \leq j} (2k-1) = \sum_{j} j^2 a_j^2 \]

and so our proof is complete.

**REFERENCE**


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