TRACE CLASS PERTURBATIONS OF ISOMETRIES
AND UNITARY DILATIONS

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ABSTRACT. Let $U$ and $V$ be isometries acting on a Hilbert space. If $U - V$ belongs to the trace class we show that the absolutely continuous parts of the corresponding minimal unitary dilations are unitarily equivalent.

Introduction. With $X$ and $Y$ complex Hilbert spaces let $\mathcal{B}(X, Y)$ denote the collection of bounded linear operators from $X$ to $Y$; the symbol $\mathcal{F}(X)$ will represent the trace class elements in $\mathcal{B}(X, X)$. If $A$ is a (not necessarily bounded) selfadjoint operator in $X$ with the spectral representation $A = \int_{\sigma} \lambda dE_{\lambda}$ and $x \in X$, then $\mu_x(\Delta) = \|E(\Delta)x\|^2$ is a nonnegative, completely additive measure defined for Borel sets $\Delta$. If this measure is absolutely continuous with respect to Lebesgue measure, we say that $x$ is absolutely continuous with respect to $A$. The collection $X(A)_{ac}$ of all vectors $x$ for which the measure $\mu_x(\cdot)$ is absolutely continuous is a closed subspace which reduces $A$, and the restriction of $A$ to this space, $A_{ac}$, is called the absolutely continuous part of $A$. A similar definition applies to a unitary operator $A$.

1. An important result on the perturbation of continuous spectra was given by Rosenblum [1] and Kato [2] and later generalized by Kuroda [3]. This theorem asserts that if $\{A, B\}$ is a pair of selfadjoint operators in a Hilbert space $X$ such that $(A - z)^{-1} - (B - z)^{-1} \in \mathcal{F}(X)$ for some $z$ with $\text{Im} \ z \neq 0$, then $A_{ac}$ is unitarily equivalent to $B_{ac}$. By taking Cayley transforms this may be rephrased in terms of unitary operators, i.e., if $U$ and $V$ are unitary operators for which $U - V \in \mathcal{F}(X)$, then $U_{ac}$ and $V_{ac}$ are unitarily equivalent. The object of this note is to prove an extension of this fact for pairs of isometries. Precisely, we prove the following:

Theorem. Let $U$ and $V$ be isometries in $X$ with minimal unitary dilations $W$ and $Z$, respectively. If $U - V \in \mathcal{F}(X)$, then $W_{ac}$ and $Z_{ac}$ are unitarily equivalent.
$U'$ and $V'$, respectively. If $U - V \in \mathcal{T}(X)$, then $U'_{ac}$ and $V'_{ac}$ are unitarily equivalent.

**Proof.** Since $U - V \in \mathcal{T}(X)$, it is certainly compact; thus it suffices to consider the case of a separable space $X$. Again, since $U - V$ is compact, a property from index theory [5] says that $\text{index}(U) = \text{index}(V)$. But since $(U - V)^{\perp} \subseteq \mathcal{R}(I)$, for any isometry $I$ on $X$, hence dimension $\mathcal{R}(U)^{\perp} = \dim \mathcal{R}(V)^{\perp}$, (the symbol \perp denotes orthogonal complement). Let $a$ equal this common value. If $a$ is infinite, then both $U_{ac}$ and $V_{ac}$ have spectral multiplicity uniformly $N$ over the entire unit circle [6, Theorem 1, p. 271] and therefore $U_{ac}$ is unitarily equivalent to $V_{ac}$. It remains to consider the case when $a$ is a finite number.

Our strategy consists in extending $U$ and $V$ to minimal unitary dilations $U'$ and $V'$ on the same dilation space $X'$ in such a manner that $U' - V' \in \mathcal{T}(X')$. The assertion of the Theorem will then follow by invoking the result of Rosenblum-Kato-Kuroda for the pair $\{U', V'\}$.

The construction uses the Wold decomposition [6, Theorem 1.1, p. 3]:

Let

\begin{equation}
U \cong S_{a} \oplus U_{u},
\end{equation}

\begin{equation}
V \cong S_{a} \oplus V_{u}
\end{equation}

($\cong$ denotes unitary equivalence) be Wold decompositions for $U$ and $V$ so that $S_{a}$ is a unilateral shift of multiplicity $a$, and \{U_{u}, V_{u}\} are unitary operators not necessarily acting on the same space. Let $h$ denote the $l_{2}$ space with dimension $a$. Without loss of generality we may assume that our original space $X$ decomposes into an orthogonal sum $X_{a} \oplus X_{(U)}$ such that $X_{a}$ is the Hilbert space whose elements are vector functions $\xi = (\xi_{0}, \xi_{1}, \xi_{2}, \cdots)$ with $\xi_{n} \in h$, $\|\xi\|^{2} = \sum_{n=0}^{\infty} \|\xi_{n}\|^{2} < \infty$. The operator $S_{a}$ acts on $X_{a}$ by $S_{a} \xi = (\eta_{0}, \eta_{1}, \eta_{2}, \cdots)$; $\eta_{0} = 0$, $\eta_{n} = \xi_{n-1}$, $n \geq 1$, and $U_{u}$ operates in $X_{(U)}$. Moreover, with $V_{u}$ operating in the space $X_{(V)}$, the decomposition (2) implies the existence of an isometric map $\phi$ of $Y \equiv X_{a} \oplus X_{(V)}$ onto $X$ such that $V = \phi(S_{a} \oplus U_{u}) \phi^{*}$, where $\phi^{*} \in \mathcal{B}(X, Y)$ is the adjoint map. Let us define dilation spaces by setting

\begin{equation}
X' = \hat{X}_{a} \oplus' X \equiv \hat{X}_{a} \oplus' X_{a} \oplus X_{(U)},
\end{equation}

\begin{equation}
Y' = \hat{Y}_{a} \oplus' Y \equiv \hat{Y}_{a} \oplus' X_{a} \oplus X_{(V)}
\end{equation}

where $\hat{X}_{a}$ is the Hilbert space whose elements are the vectors $\xi = (\cdots, \xi_{-3}, \xi_{-2}, \xi_{-1})$ with $\xi_{n} \in h$, $\|\xi\|^{2} = \sum_{n=1}^{\infty} \|\xi_{n}\|^{2} < \infty$, and $\oplus'$ denotes orthogonal direct sum. A minimal unitary dilation of $U$ is then given by the
operator $U' \in \mathcal{B}(X', X')$ defined as follows [6, Chapter 1, §2]: If

$$x' = (\cdots, \xi_{-3}, \xi_{-2}, \xi_{-1}) \oplus' (\xi_0, \xi_1, \xi_2, \cdots) \oplus x_u,$$

set

$$U'x' = (\cdots, \xi_{-4}, \xi_{-3}, \xi_{-2}) \oplus' (\xi_{-1}, \xi_0, \xi_1, \cdots) \oplus U_x'x_u.$$

Further, we construct a minimal unitary dilation $V'$ of $V$ by setting $V' = \phi'M'\phi'^*$ where $M' \in \mathcal{B}(Y', Y')$; $\phi' \in \mathcal{B}(Y', X')$, $\phi'^* \in \mathcal{B}(X', Y')$ are defined by

$$M'y' = (\cdots, \xi_{-4}, \xi_{-3}, \xi_{-2}) \oplus' (\xi_{-1}, \xi_0, \xi_1, \cdots) \oplus V'_y u,$$

$$\phi'y' = (\cdots, \xi_{-3}, \xi_{-2}, \xi_{-1}) \oplus' \phi((\xi_0, \xi_1, \xi_2, \cdots) \oplus y_u)$$

for

$$y' = (\cdots, \xi_{-3}, \xi_{-2}, \xi_{-1}) \oplus' (\xi_0, \xi_1, \xi_2, \cdots) \oplus y_u$$

and $\phi'^*$ is the adjoint of $\phi'$. We shall now prove that $U' - V' \in \mathcal{T}(X')$. For convenience we introduce the following operators: With

$$x' = \xi^{(1)} \oplus' \xi^{(2)} \oplus x \in X'; \quad \xi^{(1)} = (\cdots, \xi_{-3}, \xi_{-2}, \xi_{-1}) \in \hat{X}_a,$$

put

$$Px' = \xi^{(2)} \oplus x \in X,$$

$$P_{-1}x' = 0 \oplus' (\xi_{-1}, 0, 0, \cdots) \oplus 0 \in X',$$

$$Q_{-1}x' = 0 \oplus' (\xi_{-1}, 0, 0, \cdots) \oplus 0 \in Y'$$

so that $P \in \mathcal{B}(X', X)$, $P_{-1} \in \mathcal{B}(X', X')$ and $Q_{-1} \in \mathcal{B}(X', Y')$. Note that $P$ is a projection, whereas $P_{-1}$ and $Q_{-1}$ are shifts followed by projections of rank $a$; in particular, $P_{-1} \in \mathcal{T}(X')$. We also introduce the injection operator of $X$ into $X'$ by setting $Ix = 0 \oplus' x \in X'$ so $I \in \mathcal{B}(X, X')$. With these definitions we shall calculate that

$$U' - V' = P_{-1} + K(U - V)P - \phi'^*Q_{-1}.$$  

Choose $x' = \xi^{(1)} \oplus' \xi^{(2)} \oplus x$ as above. We then have

$$(U' - V')x' = U'x' - V'x' = U'x' - \phi'M'\phi'^*x'$$

where

$$\eta^{(1)} = (\cdots, \xi_{-4}, \xi_{-3}, \xi_{-2}), \quad \eta^{(2)} = (\xi_{-1}, \xi_0, \xi_1, \cdots).$$
Observe that
\[ \eta^{(1)} \oplus \eta^{(2)} \oplus U_u x = \eta^{(1)} \oplus r^{(1)} \oplus 0 + 0 \oplus U(\xi^{(2)} \oplus x) \]
where \( r^{(1)} = (\xi_{-1}, 0, 0, \cdots) \). Also,
\[ \phi' M [\xi^{(1)} \oplus \phi^*(\xi^{(2)} \oplus x)] = \phi' [\eta^{(1)} \oplus 0 \oplus 0 + 0 \oplus r^{(1)} \oplus 0] \\
+ \phi' M [0 \oplus \phi^*(\xi^{(2)} \oplus x)]. \]
But
\[ \phi'(\eta^{(1)} \oplus 0 \oplus 0) = \eta^{(1)} \oplus 0 \oplus 0 \in X', \]
\[ \phi'(0 \oplus r^{(1)} \oplus 0) = 0 \oplus \phi(r^{(1)} \oplus 0) \in X', \]
\[ \phi' M [0 \oplus \phi^*(\xi^{(2)} \oplus x)] = 0 \oplus \phi'(\xi^{(2)} \oplus x) \in X'. \]
Thus,
\[ (U' - V') x' = P_{-1} x' + I(U - V) P x' - \phi' Q_{-1} x' \]
and therefore
\[ U' - V' = P_{-1} + I(U - V) P - \phi' Q_{-1} \]
as operators on \( X' \). By what has been noted above, \( P_{-1} \) and \( \phi' Q_{-1} \in \mathcal{F}(X') \); since \( U - V \in \mathcal{F}(X) \) by hypothesis, it follows that \( I(U - V) P \in \mathcal{F}(X') \). We may therefore conclude that \( U' - V' \in \mathcal{F}(X') \). The result of Rosenblum-Kato-Kuroda mentioned previously now allows us to assert that \( U'_{ac} \) is unitarily equivalent to \( V'_{ac} \). This completes the proof.

**Corollary 1.** Let \( U \) and \( V \) be isometries on \( X \) with the Wold decompositions \( U \cong S_a \oplus U_u \) and \( V \cong S_a \oplus V_u \). If \( U - V \in \mathcal{F}(X) \) and \( \alpha \) is a finite number, then \( U_{ac} \) and \( V_{ac} \) are unitarily equivalent.

**Proof.** From the Theorem it follows that the dilations \( U' \), \( V' \) have unitarily equivalent absolutely continuous parts. If \( B_\alpha \) is the bilateral shift of multiplicity \( \alpha \), then
\[ U_{ac} = (B_\alpha \oplus U_u)_{ac} \cong B_\alpha \oplus U_{ac} \]
and
\[ V_{ac} = (B_\alpha \oplus V_u)_{ac} \cong B_\alpha \oplus V_{ac}. \]
Since \( \alpha \) is finite we deduce that \( U_{ac} \cong V_{ac} \).

Does the conclusion of this corollary remain valid without assuming that \( \alpha \) is finite? We do not know the answer to this question. However, we can prove the following result.
Proposition. Let $U$ and $V$ be isometries on $X$ such that $U - V$ has finite rank. Then $U_{uac}$ and $V_{uac}$ are unitarily equivalent.

Proof. Let $U - V$ have the Schmidt series $\sum_{n=1}^{d} \phi_n \otimes \psi_n$ where $d = \dim R(U - V)$ is finite. Let $\mathcal{M}$ denote the smallest (closed) subspace of $X$ which is invariant for $U$ and contains the vectors $\phi_1, \phi_2, \ldots, \phi_d$. Then $\mathcal{M}$ is also invariant for $V$. Hence, $\mathcal{M}^\perp$ is invariant for $U^*$ and $V^*$.

Moreover, the restriction operators on $\mathcal{M}^\perp$ induced by $U^*$ and $V^*$ are equal. It then follows that their unitary parts, if any, are also equal. Hence, $U$ and $V$ have the same unitary parts on $\mathcal{M}^\perp$. On the other hand, the operators $U$ and $V$ restricted to $\mathcal{M}$ have index $\geq -d$ and so by Corollary 1 we may conclude that their unitary parts have the same absolutely continuous pieces. Now, by the Wold decomposition it is clear that a vector $x$ belongs to the unitary part of an isometry $I$ provided $\|I^n x\| = \|x\|$ ($n = 1, 2, \ldots$). Suppose $y$ is such a vector for $U$ which is orthogonal to the subspaces of $\mathcal{M}$ and $\mathcal{M}^\perp$ on which $U$ is unitary. Set $y = y_\mathcal{M} \oplus y_{\mathcal{M}^\perp}$ with $y_\mathcal{M} \in \mathcal{M}$ and $y_{\mathcal{M}^\perp} \in \mathcal{M}^\perp$. By virtue of the Wold decomposition it follows that $\|U_n^* y_\mathcal{M}\| \to 0$ and $\|U_n^* y_{\mathcal{M}^\perp}\| \to 0$. Hence, $y = 0$. This concludes the proof.

Various authors [7], [8] have shown that if $S$ is the simple unilateral shift and $U$ is any unitary operator, then there exists a compact operator $K$ such that $S + K$ is unitarily equivalent to $S \oplus U$. On the other hand the present author and J. Pincus [9] have recently shown that if $U$ is any unitary operator with purely singular spectra, then there exists a trace class operator $K$ such that $S + K$ is unitarily equivalent to $S \oplus U$.

REFERENCES


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