NOTE ON A VOLterra INTEGRO-DIFFERENTIAL EQUATION SYSTEM

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ABSTRACT. This note concerns a type of nonlinear Volterra integro-differential equation system which has only asymptotically zero solutions. It is a simple generalization of a one-dimensional result of J. J. Levin. It is a correction to another such note and gives a counterexample to that note.

In [1] Kemp discusses the integro-differential equation

\[ x'(t) = - \int_0^t a(t - w) g(x(w)) \, dw \]

where \( x \) maps \([0, \infty)\) into \( \mathbb{R}^n \), \( g \) maps \( \mathbb{R}^n \) into \( \mathbb{R}^n \) and \( a \) maps \([0, \infty)\) into \( n \times n \) matrices over \( \mathbb{R} \). Take the following as basic assumptions:

(a) \( g \in C(\mathbb{R}^n) \), \( x^T g(x) > 0 \) for \( x \neq 0 \) (where \( x^T \) is the transpose of the \( n \times 1 \) matrix \( x \)), there exists a scalar function \( G \in C^1(\mathbb{R}^n) \) such that \( g \) is the gradient of \( G \) and \( G(x) \to \infty \) as \( |x| \to \infty \);

(b) \( a \in C([0, \infty)) \), and \((-1)^k a^{(k)}(t)\) is a real symmetric positive semidefinite matrix for \( 0 < t < \infty \), \( k = 0, 1, 2, 3 \).

Under (a) and (b), Kemp states the following theorem about asymptotic behavior.

**Theorem I.** Any solution \( u(t) \) of (1) satisfies

\[ \lim_{t \to \infty} u^{(j)}(t) = 0 \quad (j = 0, 1, 2) \]

provided that \( a(t) \neq a(0) \).

However the system below is a counterexample to that theorem.

Received by the editors November 1, 1972 and, in revised form, September 26, 1973.

**AMS (MOS) subject classifications (1970).** Primary 45M05, 45J05; Secondary 45G99, 45D05.

**Key words and phrases.** Integro-differential equation, nonlinear Volterra equation, integral equation system, asymptotic behavior.
\[
\begin{bmatrix}
    u_1'(t) \\
    u_2'(t)
\end{bmatrix} = -\int_0^t \begin{bmatrix}
    \exp(-(t-w)) & 0 \\
    0 & 0
\end{bmatrix} \begin{bmatrix}
    u_1(w) \\
    u_2(w)
\end{bmatrix} dw, \quad u_1(0) = 0; \quad u_2(0) = 1.
\]

Kemp's result is an attempt at generalizing a one-dimensional theorem of J. J. Levin which is in [2]. In the one-dimensional argument the following result is important (Lemma 2 in [2]).

**Lemma II.** If (b) is satisfied and if \( a(t) \neq a(0) \) then either \(-a'(t), a''(t) > 0 \) for \( 0 < t < \infty \) or there exists a \( t_0 > 0 \) such that \(-a'(t), a''(t) > 0 \) for \( 0 < t < t_0 \) and \( a(t) = a(t_0) = a(\infty) > 0 \) for \( t_0 \leq t < \infty \).

Levin uses this lemma to show that there is an \( S > 0 \) such that \( a''(t) > 0 \) for \( 0 < t \leq S \). The lemma, however, does not hold for the \( n \)-dimensional case. To generalize Levin's theorem we may use Lemma III.

**Lemma III.** If (b) is satisfied and if there is no nonzero \( x \in \mathbb{R}^n \) such that \( x^T a(t)x = x^T a(0)x \) for all \( t, 0 \leq t < \infty \), then there exists an \( S > 0 \) such that \( a''(t) \) is positive definite for \( 0 < t \leq S \).

With Lemma III all of Levin's arguments may be put through, with a little extra work, to give Theorem IV.

**Theorem IV.** If (a) and (b) hold and \( u(t) \) is any solution of (1), then \( u(t) \) satisfies (2) provided that there is no nonzero \( x \in \mathbb{R}^n \) such that \( x^T a(t)x = x^T a(0)x \) for all \( t, 0 \leq t < \infty \).

**Proof of Lemma III.** The function \( x^T a(t)x, x \neq 0 \), satisfies the lemma of Levin. Therefore for any \( x \in \mathbb{R}^n, x \neq 0 \), there exists \( t(x) \) such that \( x^T a''(t)x > 0 \) for \( 0 < t < t(x) \). Note that \( a''(t) \) is positive definite if and only if for all \( x \in \mathbb{R}^n, |x| = 1, x^T a''(t)x > 0 \).

For a contradiction proof assume that there exist a sequence \( x_n, |x_n| = 1 \), and a sequence \( t_n, t_n > 0 \) and \( \lim t_n = 0 \), such that \( x_n^T a''(t_n)x_n = 0 \). Without loss of generality assume \( \lim x_n = y \) with \( |y| = 1 \). Choose \( T \) such that \( 0 < T < t(y) \) so that \( y^T a''(T)y > 0 \). \( \lim x_n^T a''(T)x_n = y^T a''(T)y > 0 \) implies there is some \( N \) such that for \( n > N, x_n^T a''(T)x_n > 0 \). \( x_n^T a''(t)x_n \) is nonincreasing in \( t \) for \( t > 0 \) and any \( n \) (just look at its derivative); hence \( x_n^T a''(T)x_n > 0 \) for \( n > N \) implies \( x_n^T a''(t)x_n > 0 \) for \( 0 < t < T \) and \( n > N \). Now if \( n \) is sufficiently big such that \( n > N \) and \( 0 < t_n < T \), we have two
contradictory statements: $x_n^T a'(t_n) x_n = 0$ and $x_n^T a''(t_n) x_n > 0$. Thus there is a contradiction and there must exist an $S$ as described.

REFERENCES
