A CHARACTERIZATION OF THE CONNECTIVITY OF A MANIFOLD IN TERMS OF LARGE OPEN CELLS

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ABSTRACT. If $k$ and $n$ are integers, $0 \leq k < n - 3$, and $M^n$ is a topological $n$-manifold without boundary, it is shown that $M$ is $k$-connected if and only if there is a "tame" $(n - k - 1)$-dimensional closed subset $X$ in $M$ such that $M - X$ is homeomorphic to $E^n$.

1. Introduction. In [1] Morton Brown proved that any compact connected topological $n$-manifold $M^n$ without boundary is the continuous image of an $n$-cell such that the boundary of the cell is taken onto a subset of $M$ having dimension $\leq n - 1$ and the interior of the cell is taken homeomorphically onto the rest of $M$. Thus any compact $n$-manifold can be obtained from euclidean $n$-space $E^n$ by simply (and carefully) pasting on a space $X$ of dimension $\leq n - 1$. We examine here the question of just how small the dimension of $X$ can be made and show that this question is directly related to the connectivity of $M$.

By a (topological) $n$-manifold $M^n$, we mean a separable metric space, each point of which has an open neighborhood homeomorphic to $E^n$ (we shall assume throughout that all manifolds are without boundary). A euclidean chart for $M$ is a pair $(h, W)$ where $W$ is an open subset of $M$ and $h : E^n \to W$ is a homeomorphism. For any chart $(h, W)$ of $M$ and any real number $t > 0$, let $W_t = h(C_t)$ where $C_t$ is the (closed) $n$-cell in $E^n$ with center 0 and radius $t$.

A closed subset $X$ of a topological space $Y$ is a $Z_1$-set if for every nonempty 1-connected open subset $U$ of $Y$, $U - X$ is nonempty and 1-connected. If $Y$ is a metric space, $X$ is a subset of $Y$, and $\epsilon > 0$, let $N(X, \epsilon)$ denote the set of points in $Y$ whose distance from $X$ is less than $\epsilon$. An $\epsilon$-push of the pair $(Y, X)$ is a homeomorphism $h$ of $Y$ for which an $\epsilon$-isotopy $H$ of $Y$ exists satisfying: $H_0 = 1$, $H_1 = h$, and $H_t|Y - N(X, \epsilon) = 1$ for each

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A closed subset $X$ of $E^n$ is said to be a strong $Z_m$-set in $E^n$ ($m$ an integer, $-1 \leq m < n$) if for each compact subpolyhedron $Q$ of $E^n$ having dimension $\leq m + 1$, and each $\epsilon > 0$, there exists an $\epsilon$-push $h$ of $(E^n, X \cap Q)$ such that $h(X) \cap Q = \emptyset$. Heuristically, one should think of a strong $Z_m$-set in $E^n$ as a "tame" subset of $E^n$ having dimension $\leq n - m - 2$: the strong $Z_m$-sets in $E^n$, $m \geq 1$, are precisely those which are $Z_1$-sets and have dimension $\leq n - m - 2$ (see 2.1 below). A closed subset $X$ of an $n$-manifold $M$ is a strong $Z_m$-set in $M$ if for each point $x$ in $X$ there is a euclidean chart $(h, W)$ for $M$ such that $x \in W$ and $h^{-1}(X)$ is a strong $Z_m$-set in $E^n$.

The main result of this paper is

**Theorem 1.1.** Let $k$ and $n$ be integers, $0 \leq k < n - 3$, and let $M^n$ be an $n$-manifold. Then $M$ is $k$-connected if and only if there is a strong $Z_{k-1}$-set $X$ in $M$ such that $M - X$ is homeomorphic to $E^n$.

Our proof in the "if" direction is basically a general position argument: A singular $k$-sphere in $M$ is pushed off $X$, and hence into $M - X$, where it bounds. The proof in the "only if" direction is similar to Morton Brown's proof in [11]. We start with an open cell in $M$ and engulf a dense $k$-dimensional subset of $M$ whose complement is $(n - k - 1)$-dimensional. The complement of the open cell (after the engulfing) is a strong $Z_{k-1}$-set in $M$. A more useful form of 1.1 is

**Theorem 1.2.** Let $M^n$ be an $n$-manifold.

1. $M$ is connected if and only if there is a closed subset $X$ of $M$ such that $\dim X \leq n - 1$ and $M - X$ is homeomorphic to $E^n$.

2. If $n \geq 4$, then $M$ is simply connected if and only if there is a closed subset $X$ of $M$ such that $\dim X \leq n - 2$ and $M - X$ is homeomorphic to $E^n$.

3. If $2 \leq k \leq n - 3$, then $M$ is $k$-connected if and only if there is a $Z_1$-set $X$ in $M$ such that $\dim X \leq n - k - 1$ and $M - X$ is homeomorphic to $E^n$.

Throughout the remainder of this paper let $M^n$ be a fixed $n$-manifold with metric $d$.

In §2 we shall discuss strong $Z_m$-sets in $M$ and in §3 we present the proofs of 1.1 and 1.2.

2. Tame sets in topological manifolds. First we give a characterization of strong $Z_m$-sets in terms of dimension and local homotopy properties.

**Lemma 2.1.** Let $m$ be an integer, $-1 \leq m < n$, and let $X$ be a closed subset of $M$.

1. If $X$ is a strong $Z_m$-set in $M$, then $\dim X \leq n - m - 2$. 
(2) If \( \dim X < n - 1 \), then \( X \) is a strong \( Z_{n-1} \)-set in \( M \).

(3) If \( \dim X < n - 2 \) and \( n \neq 3 \), then \( X \) is a strong \( Z_0 \)-set in \( M \).

(4) If \( \dim X = n - m - 2 < n - 3 \), \( n \neq 4 \), and \( X \) is a \( Z_1 \)-set in \( M \), then \( X \) is a strong \( Z_{m-1} \)-set in \( M \).

**Proof.** For the case \( M = E^n \), this lemma is a precise restatement of 3.1 of [5]. To prove the lemma for an arbitrary manifold, one need only look at charts and apply the case \( M = E^n \).

**Lemma 2.2.** Let \( X \) be a strong \( Z_m \)-set in \( M \) and let \( Q \) be a compact \((m + 1)\)-dimensional polyhedron. Then any map \( f: Q \to M \) is homotopic to a map \( g: Q \to M - X \).

**Proof.** There exist finitely many euclidean charts \((h_1, W_1), \ldots, (h_r, W_r)\) such that \( f(Q) \subseteq \bigcup_{i=1}^r W_i \) and \( h_i^{-1}(X) \) is a (possibly empty) strong \( Z_m \)-set in \( E^n \). Furthermore, there exists a real number \( t \) such that \( f(Q) \subseteq W_{1t} \cup \cdots \cup W_{rt} \) where \( W_{it} = h_i(C_t) \) for each \( i \leq r \). For each \( i \leq r \), let \( Q_i \) be a compact subpolyhedron of \( Q \) such that \( f(Q_i) \subseteq W_i \cup \cdots \cup W_{it} \) where \( W_{it} = h_i(h(Q)) \) for each \( i \). We shall construct, by induction, a sequence of maps \( f_1, \ldots, f_r \) from \( Q \) into \( M \) such that

(1) \( f \) is homotopic to \( f_1 \), and \( f_i \) is homotopic to \( f_{i+1} \) for each \( i \leq r \),

(2) \( f_i(Q_j) \subseteq W_j \) for each \( i, j \leq r \), and

(3) \( f_i(Q_1 \cup \cdots \cup Q_i) \cap X = \emptyset \) for each \( i \leq r \).

Then \( g = f_r \) is a map homotopic to \( f \) and \( g(Q) \cap X = \emptyset \). To start the induction consider the map \( f_1: Q_1 \to W_1 \). By the simplicial approximation theorem there is a map \( f_1': Q \to M \) such that \( f_1' \) is homotopic to \( f \), \( f_1'(Q_j) \subseteq W_j \) for each \( j \leq r \), and \( f_1'|Q_1: Q_1 \to W_1 \) is PL where \( W_1 \) has the PL structure induced by \( h_1 \). Since \( h_1^{-1}(X) \) is a strong \( Z_m \)-set in \( E^n \), there is a homeomorphism \( g_1: M \to M \) isotopic to the identity such that \( g_1 f_1'(Q_j) \subseteq W_j \) for each \( j \leq r \) and \( g_1 f_1'(Q_1) \cap X = \emptyset \). Then \( f_1 = g_1 f_1' \) is homotopic to \( f \), \( f_1(Q_j) \subseteq W_j \) for each \( j \leq r \), and \( f_1(Q_1) \cap X = \emptyset \). Now suppose that \( f_{i-1} \) has been chosen, \( i \leq r \), and consider the map \( f_{i-1}|Q_i: Q_i \to W_i \). By the simplicial approximation theorem there is a map \( f_i': Q \to M \) such that \( f_i' \) is homotopic to \( f_{i-1} \), \( f_i'(Q_j) \subseteq W_j \) for each \( j \leq r \), \( f_i'(Q_1 \cup \cdots \cup Q_{i-1}) \cap X = \emptyset \), and \( f_i'|Q_i: Q_i \to W_i \) is PL where \( W_i \) has the structure induced by \( h_i \). Since \( h_i^{-1}(X) \) is a strong \( Z_m \)-set in \( E^n \), there is a homeomorphism \( g_i: M \to M \) isotopic to the identity, such that \( g_i f_i'(Q_j) \subseteq W_j \) for each \( j \leq r \), \( g_i \) is fixed on \( f_i'(Q_1 \cup \cdots \cup Q_{i-1}) \), and \( g_i f_i'(Q_1) \cap X = \emptyset \). Then \( f_i = g_i f_i' \) is homotopic to \( f \), \( f_i(Q_j) \subseteq W_j \) for each \( j \leq r \), and \( f_i(Q_1 \cup \cdots \cup Q_i) \cap X = \emptyset \).

We now construct a \( k \)-dimensional dense subset of \( E^n \). Let \( k \) be an integer, \( 0 \leq k \leq n \), and let \( f_0 \) be a rectilinear PL triangulation of \( E^n \) such that all the \( n \)-simplexes of \( f_0 \) have the same diameter. For each integer
i ≥ 1, let \( J_i \) be the \( i \)th barycentric subdivision of \( J_0 \) and let \( J^k_i \) be the \( k \)-skeleton of \( J_i \). Finally, set \( \overline{B}_n^k = \bigcup_{i=1}^{\infty} |J^k_i| \) and \( \overline{P}_{n-k-1} = E^n - \overline{B}_n^k \).

Clearly \( \overline{B}_n^k \) is \( k \)-dimensional and \( \overline{P}_{n-k-1} \) is \((n - k - 1)\)-dimensional. Moreover, \( \overline{B}_n^k \) satisfies a very nice "absorption" property:

**Lemma 2.3.** Let \( Q \) be a compact \( k \)-dimensional subpolyhedron of \( E^n \), let \( U \) be an open subset of \( E^n \), and let \( \epsilon > 0 \). Then there is a homeomorphism \( h: E^n \to E^n \), fixed outside \( U \) and moving points a distance less than \( \epsilon \), such that \( h(Q \cap U) \subset \overline{B}_n^k \).

**Proof.** This follows directly from Lemma 4.5 of [5].

**Lemma 2.4.** A closed subset \( X \) of \( E^n \) which is contained in \( \overline{P}_{n-k-1} \) is a strong \( Z_{k-1} \)-set in \( E^n \).

**Proof.** Let \( Q^k \) be a compact \( k \)-dimensional subpolyhedron of \( E^n \), let \( U \) be an open set in \( E^n \) containing \( X \cap Q \), and let \( \epsilon > 0 \). By 2.3, there is a homeomorphism \( h \), fixed outside \( U \) and moving points a distance less than \( \epsilon \), such that \( h(Q \cap U) \subset \overline{B}_n^k \). In particular, \( h^{-1}(X) \cap Q = \emptyset \). Hence if \( h^{-1} \) were \( \epsilon \)-isotopic to the identity by an isotopy fixing \( E^n - U \), then \( X \) would be a strong \( Z_{k-1} \)-set. But the existence of such an isotopy follows easily from the results in [3].

We now construct a dense \( k \)-dimensional subset of \( M \). Let \( \{(h_i, W_i)\} \) be a countable collection of euclidean charts such that \( M = \bigcup_{i=1}^{\infty} W_i \) and let \( \overline{B}_M^k = \bigcup_{i=1}^{\infty} h_i(\overline{B}_n^k) \) and \( \overline{P}_M^{n-k-1} = M - \overline{B}_M^k \). Clearly \( \overline{B}_M^k \) is \( k \)-dimensional and \( \overline{P}_M^{n-k-1} \) is \((n - k - 1)\)-dimensional. Moreover, \( \overline{B}_M^k \) can be written as the countable union of compact subsets \( \{\overline{B}_{(i)}^k\}_{i=1}^{\infty} \) of \( M \) having the following property: if \( i > 1 \), then there exists \( j > 1 \) such that \( \overline{B}_{(i)}^k \subset W_i \), and \( B_{(i)}^{-1}(\overline{B}_{(i)}^k) \) is a compact subpolyhedron of \( E^n \) having dimension \( \leq k \). For the remainder of this paper we fix \( \{(h_i, W_i)\}, \overline{B}_M^k, \overline{P}_M^{n-k-1}, \) and \( \overline{B}_{(i)}^k \) as above.

The final lemma of this section follows directly from 2.4.

**Lemma 2.5.** A closed subset of \( M \) which is contained in \( \overline{P}_M^{n-k-1} \) is a strong \( Z_{k-1} \)-set in \( M \).

3. Connectivity in topological manifolds. Let \( X \) be a \( Z_{k-1} \)-set in \( M \) where \( 0 \leq k < n \) and let \( f: S^k \to M \) be a map of the \( k \)-sphere into \( M \). By Lemma 2.2, \( f \) is homotopic to a map \( g: S^k \to M - X \). If \( M - X \) is homeomorphic to \( E^n \), then \( g \) extends to a map of the \((k + 1)\)-ball into \( M - X \) and hence \( f \) is null-homotopic in \( M \). This proves

**Theorem 3.1.** If there is a strong \( Z_{k-1} \)-set \( X \) in \( M^n \), \( 0 < k < n \), such that \( M - X \) is homeomorphic to \( E^n \), then \( M \) is \( k \)-connected.
To prove the converse of 3.1 (for codimension \( \geq 3 \)), we require the following engulfing lemma; its proof is almost precisely the same as the proof of Lemma 1 of [2] and therefore we leave the details to the reader.

**Lemma 3.2.** Let \( k \) be an integer, \( 0 \leq k \leq n - 3 \), and let \( M^n \) be \( k \)-connected. Let \( Q \) be a compact subset of \( M \) such that for some chart \((g, U)\) of \( M \), \( Q \subset U \) and \( g^{-1}(Q) \) is a \( k \)-dimensional subpolyhedron of \( E^n \). If \((h, W)\) is a euclidean chart for \( M \) and \( t \) is a positive real number, then there is a homeomorphism \( f \) of \( M \) such that \( f|_{W_t} = 1 \) and \( f(W) \supset Q \).

**Lemma 3.3.** If \( k \) is an integer, \( 0 \leq k \leq n - 3 \), and \( M^n \) is \( k \)-connected, then there is a euclidean chart \((g, U)\) of \( M \) such that \( U \) contains \( \overline{B}^k_{M} \).

**Proof.** Consider the compact subsets \( \overline{B}^k_{M(i)} \) of \( M \) as defined in the previous section and let \((h, W)\) be any euclidean chart of \( M \). By Lemma 3.2, there is a homeomorphism \( f_1 \) of \( M \) such that \( f_1(W) \supset \overline{B}^k_{(1)} \). Since \( \overline{B}^k_{1} \) is compact, there is a real number \( t_1 \geq 1 \) such that \( f_1(W_{t_1}) \supset \overline{B}^k_{(1)} \). Applying the lemma again, there is a homeomorphism \( f_2 \) of \( M \) such that \( f_2/_{f_1(W_{t_1})} = 1 \) and \( f_2f_1(W) \supset \overline{B}^k_{(2)} \). Let \( t_2 \geq \max \{t_1, 2\} \) and such that \( f_2f_1(W_{t_2}) \supset \overline{B}^k_{(2)} \). Continuing inductively, there is a sequence \( \{f_i\} \) of homeomorphisms of \( M \) and a sequence \( t_i \leq 2 \leq \cdots \) of real numbers such that

1. \( f_i|_t f_{i-1} \cdots f_1(W_{t_{i-1}}) = 1 \),
2. \( f_i \cdots f_1(W_{t_i}) \supset \overline{B}^k_{(i)} \), and
3. \( t_i \geq i \)

for each \( i \geq 1 \). Then \( f = \lim_{i \to \infty} f_i \cdots f_1|W \) is an embedding of \( W \) into \( M \) such that \( f(W) \supset \overline{B}^k_{M} \) and hence \((g, U) = (f h, f(W))\) is a euclidean chart satisfying the desired condition.

**Theorem 3.4.** If \( k \) is an integer, \( 0 \leq k \leq n - 3 \), and \( M^n \) is \( k \)-connected, then there is a strong \( Z_{k-1} \)-set \( X \) in \( M \) such that \( M - X \) is homeomorphic to \( E^n \).

**Proof.** Let \((g, U)\) be a chart in \( M \) with \( \overline{B}^k_{M} \subset U \). Then \( X = M - U \) is closed in \( M \) and is contained in \( \overline{P}^{n-k-1} \). By 2.5, \( X \) is a strong \( Z_{k-1} \)-set in \( M \).

**Proof of Theorem 1.1.** Apply 3.1 and 3.4.

**Proof of Theorem 1.2.** (1) If \( M \) is connected, the result follows from the proof of the Theorem in [1]. The proof given there is an elementary form of engulfing and does not require codimension three. The converse is trivial since 0-spheres can easily be pushed off dimension 1 subsets. The proofs of (2) and (3) follow easily from 3.4 and 2.4.

We end the paper with a rather strange corollary to 3.3. While not direct-
ly related to the above results, it does show that the subset $\overline{B}^k_M$ of $M$ is independent of the charts $\{(h_i, W_i)\}$ and, in fact, depends only on the connectivity of $M$.

**Corollary 3.5.** Let $k$ be an integer, $0 \leq k \leq n - 3$, $n \neq 4$, and let $M$ be $k$-connected. Then $\overline{B}^k_M$ and $\overline{B}^k_n$ are homeomorphic.

**Proof.** Let $(g, U)$ be a chart in $M$ such that $\overline{B}^k_M \subseteq U$. By 5.4 and 2.5 of [5], there is a homeomorphism of $E^n$ which takes $g^{-1}(\overline{B}^k_M)$ onto $\overline{B}^k_n$.

**REFERENCES**


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