SEPARABLE MENDER-REGULAR HAUSDORFF CURVES ARE METRIZABLE

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ABSTRACT. It is shown that every connected, separable, locally compact and locally peripherally finite Hausdorff space is metrizable.

1. Introduction. A Menger-regular Hausdorff space is a connected Hausdorff topological space with a base for the topology consisting of neighborhoods with finite boundaries. We prove the following:

Theorem 1. If \( M \) is a Menger-regular Hausdorff continuum and \( A \) is a closed subset of \( M \), and \( S \) is a dense subset of \( M \), then there is a subset \( T \) of \( A \) which is dense in \( A \) such that the cardinality of \( T \) is less than or equal to the cardinality of \( S \).

Corollary 1.1. If \( M \) is a separable Menger-regular Hausdorff continuum, every arc contained in \( M \) is metrizable.

Theorem 2. Every separable Menger-regular Hausdorff continuum is metrizable.

Corollary 2.1. Every separable locally compact Menger-regular Hausdorff space is metrizable.

Suppose \( M \) is a Menger-regular Hausdorff continuum. Using techniques directly analogous to those for metric continua, it can be shown that each two closed disjoint subsets of \( M \) are separated in \( M \) by a finite subset of \( M \) and that \( M \) is locally connected [1, p. 96], [2, p. 107]. Obviously each subcontinuum of \( M \) is a Menger-regular Hausdorff continuum and therefore is locally connected. If \( O \) is a connected open subset of \( M \), and \( P \) and \( Q \) belong to \( O \), there is in \( O \) an irreducible continuum \( K \) from \( P \) to \( Q \). Since \( K \) is locally connected, each point of \( K \) distinct from \( P \) and \( Q \) is a cutpoint.

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of $K$ and $K$ is therefore a Hausdorff arc. Consequently $M$ is Hausdorff arcwise connected and locally Hausdorff arcwise connected. If $P$ is a point of $M$ and $C$ is a closed subset of $M$, an arc from $P$ to $C$ in $M$ is $\{P\}$ if $P$ belongs to $C$ and if $P$ does not belong to $C$, it is a Hausdorff arc with $P$ as one endpoint, its other endpoint in $C$ and which contains no other point of $C$. We let $N$ denote the positive integers.

2. Proof of Theorem 1. Suppose the hypothesis. Let $\omega$ denote the family of all $W$ such that $W$ is a disjoint collection each member of which is an arc from a point of $S$ to $A$. Then $\omega$ is partially ordered by containment, the union of any chain of elements of $\omega$ is a member of $\omega$ which is an upper bound of that chain, and therefore there is a maximal element $W_{\text{max}}$ of $\omega$.

Let $T = A \cap \bigcup W_{\text{max}}$. The cardinality of $T$ is obviously less than or equal to the cardinality of $S$. We will show that $T$ is dense in $A$.

Suppose $T$ is not dense in $A$ and that $P$ is a point of $A - \text{Cl}(T)$. There is a finite set $F$ separating $P$ from $\text{Cl}(T)$ in $M$, and because the members of $W_{\text{max}}$ are disjoint, $F$ intersects at most a finite number of them. Consequently, $P$ is not a limit point of $\bigcup W_{\text{max}}$ and there exists a connected open set $O$ that contains $P$ and misses $\bigcup W_{\text{max}}$. Then $O$ contains a point $Q$ of $S$ and also contains an arc $\alpha$ from $Q$ to $A$. But $W_{\text{max}} \cup \{\alpha\}$ is a member of $\omega$ which properly contains $W_{\text{max}}$ and this is a contradiction.

Proof of Corollary 1.1. If $\alpha$ is an arc in $M$ and $M$ is separable, then, from Theorem 1, $\alpha$ is separable, and it is known that every separable Hausdorff arc is metrizable [2, p. 31].

3. Proof of Theorem 2. Suppose $M$ is a separable Menger-regular Hausdorff continuum and $\{P_0, P_1, P_2, \ldots\}$ is a countable sequence of distinct points which is dense in $M$. We define inductively the sequence $\{A_0, A_1, A_2, \ldots\}$.

(i) Let $A_0$ denote $\{P_0\}$.

(ii) Suppose $n$ is a nonnegative integer and $A_i$ has been defined for $0 \leq i \leq n$. Let $A_{n+1}$ denote an arc from $P_{n+1}$ to $\bigcup \{A_i \mid i = 0, 1, \ldots, n\}$.

For each $n \geq 0$, let $D_n$ denote $\bigcup \{A_i \mid i = 0, 1, \ldots, n\}$, and let $Q_{n+1}$ denote the point of $A_{n+1}$ that belongs to $D_n$. Then for $n \in N$, $D_n$ is the union of a finite number of metric arcs and is itself a metric space, in fact, a metric dendron (locally connected metric continuum which contains no simple closed curve). For $n \geq 0$, define $\rho_{n+1} : D_{n+1} \to D_n$ by

$$\rho_{n+1}(x) = x \quad \text{if } x \in D_n,$$

$$\rho_{n+1}(x) = Q_{n+1} \quad \text{if } x \in A_{n+1} \setminus D_n.$$
It is apparent that, for \( n \in \mathbb{N} \), \( \rho_n \) is continuous, and we let \( D_\infty \) denote the inverse limit space of the inverse limit system \( \{ D_n, \rho_n, n \in \mathbb{N} \} \). Since, for \( n \in \mathbb{N} \), \( D_n \) is a compact metric continuum, \( D_\infty \) is a compact metric continuum. It can be shown that \( D_\infty \) is a dendron, but we do not require this.

Suppose \(( x_1, x_2, \cdots) \) belongs to \( D_\infty \). For \( n \in \mathbb{N} \), either \( x_{n+1} \) belongs to \( D_n \), in which case \( x_{n+1} = x_{n'} \) or \( x_{n+1} \) does not belong to \( D_n \), in which case \( x_{n+1} \) belongs to \( A_{n+1} \) and \( x_n \) is \( Q_{n+1} \). In the first case we let \([ x_{n'}, x_{n+1} ] \) denote \([ x_n ] \), and in the second case we let \([ x_{n'}, x_{n+1} ] \) denote the subarc of \( A_{n+1} \) with endpoints \( x_n \) and \( x_{n+1} \). For \( m, n \in \mathbb{N}, m < n \), we let \([ x_m, x_n ] \) denote \( \bigcup \{ [x_k, x_{k+1}] | k = m, n - 1 \} \). By induction arguments which we omit, it can be shown that if \( i, j, k \in \mathbb{N}, i < j < k \), then

(a) \([ x_i, x_j ] \) is either \([ x_i ] \) or an arc with endpoints \( x_i \) and \( x_j \).

(b) \( x_i \) is the only point of \([ x_i, x_j ] \) which belongs to \( D_i \).

(c) \( x_j \) is the only point common to \([ x_i, x_j ] \) and \([ x_i, x_k ] \).

Lemma 1. If \(( x_1, x_2, \cdots) \) is a point of \( D_\infty \), there is a point \( P \) of \( M \) such that \( \lim [ x_n, x_{n+1} ] = \{ P \} \).

Proof. The sequence \( x_1, x_2, \cdots \) has at least one cluster point, \( P \). Suppose \( q_1, q_2, \cdots \) is a sequence such that, for \( n \in \mathbb{N} \), \( q_n \in [ x_n, x_{n+1} ] \), and \( q_1, q_2, \cdots \) has a cluster point \( Q \) distinct from \( P \). There is a finite subset \( F \) of \( M \) such that \( M - F = U \cup V \), \( U \) and \( V \) disjoint open subsets of \( M \) containing \( P \) and \( Q \) respectively. There is an infinite increasing sequence \( n(1), n(2), \cdots \) such that for \( k \in \mathbb{N} \), \( x_{n(2k-1)} \) belongs to \( U \) and \( q_{n(2k)} \) belongs to \( V \). Then for \( k \in \mathbb{N} \), \( F \) contains a cutpoint of \( [ x_{n(2k-1)}, x_{n(2k+1)} ] \), and because \( F \) is finite, there is a point \( f \) of \( F \) and there are two integers \( i \) and \( j \) such that \( f \) is a cutpoint of both \( [ x_{n(2i-1)}, x_{n(2i+1)} ] \) and \( [ x_{n(2j-1)}, x_{n(2j+1)} ] \). This and statement (c) above lead to a contradiction. It follows that \( \{ P \} \) is \( \lim [ x_n, x_{n+1} ] \).

We define \( \phi: D_\infty \rightarrow M \) by: If \( x = ( x_1, x_2, \cdots) \) belongs to \( D_\infty \), \( \phi(x) = \lim x_n \).

It follows from Lemma 1 that \( \phi \) is a function. We show that

Lemma 2. \( \phi \) is continuous.

Proof. Suppose \( x \in D_\infty \) and \( O \) is an open set in \( M \) containing \( \phi(x) \). There is an open set \( R \) in \( M \) which contains \( \phi(x) \) such that \( \text{Cl}(R) \) is a subset of \( O \) and \( R \) has a finite boundary \( F \). Since \( F \) is finite, there is \( i \in \mathbb{N} \) such that \( D_i \) contains \( F \cap \bigcup \{|D_n| | n \in \mathbb{N} \} \). There is an integer \( j \) greater than \( i \) such that \( x_j \) belongs to \( R \) and there is a connected open subset \( U_{x_j} \) of \( D_j \) that contains \( x_j \) and is a subset of \( R \). Let \( U \) denote the
points $y$ of $D_{\infty}$ such that the $j$th coordinate of $y$ belongs to $U_j$. Then $U$ is an open subset of $D_{\infty}$ which contains $y$ and we next show that $\phi(U) \subseteq \text{Cl}(R) \subseteq O$.

Suppose $y = (y_1, y_2, \cdots)$ is a point of $U$ and $\phi(y)$ does not belong to $\text{Cl}(R)$. Then $y_j$ belongs to $U_j$ and therefore belongs to both $D_j$ and $R$. But since $\phi(y) = \lim y_n$ does not belong to $\text{Cl}(R)$, there is an integer $k > j$ such that $y_k$ does not belong to $\text{Cl}(R)$. But then $F$ must contain a point $y'$ of $[y_j, y_k] \subseteq D_k$, and since $F \cap D_k \subseteq D_i \subseteq D_j$, $y'$ belongs to $D_j$. But $y_j$ and $y'$ are two points of $[y_j, y_k]$ that belong to $D_j$ and this is a contradiction to (b) above.

Lemma 3. $\phi$ maps $D_{\infty}$ onto $M$.

For $n \in N$, $\phi$ maps the point $(\rho_1 \cdots \rho_{n-1}(P_n), \rho_{n-1}(P_n), \cdots, \rho_{n-1}(P_n), P_n, P_n, \cdots)$ onto $P_n$. Therefore $\phi(D_{\infty})$ is dense in $M$ and since $D_{\infty}$ is compact, $\phi(D_{\infty})$ is compact and is therefore $M$.

We have shown that $M$ is the continuous image of a compact metric space, and since $M$ is Hausdorff, it follows that $M$ is metrizable.

Proof of Corollary 2.1. Suppose $M$ is a separable locally compact Menger-regular Hausdorff space. The one point compactification of $M$ is a separable Menger-regular Hausdorff continuum, which by Theorem 2 is metrizable, and it follows that $M$ is metrizable.

REFERENCES


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