MULTIPLIERS VANISHING AT INFINITY
FOR CERTAIN COMPACT GROUPS

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ABSTRACT. We prove for certain compact groups $G$ and $1 < p < \infty$, $p \neq 2$, that there exist operators commuting with left translations on $L^p(G)$ which are compact as operators on $L^2(G)$ but not as operators on $L^p(G)$.

Let $G$ be a compact group and let $\Gamma$ be the dual object of $G$, that is, the set of equivalence classes of irreducible unitary representations of $G$. For each $\gamma \in \Gamma$ we fix a representative $D_\gamma$ of $\gamma$ and a Hilbert space $\mathbb{H}_\gamma$ of dimension $d_\gamma$ on which $D_\gamma(x)$ acts. With this notation, if $f \in L^1(G)$ we can write the Fourier series of $f$ as

$$f(x) \sim \sum_{\gamma \in \Gamma} d_\gamma \text{tr} (\hat{f}(\gamma) D_\gamma(x))$$

where $\text{tr}$ is the ordinary trace and

$$\hat{f}(\gamma) = \int_G f(x) D_\gamma(x^{-1}) \, dx$$

is a linear transformation acting on $\mathbb{H}_\gamma$. (Warning: the notation here is not the same as in [7]: $\hat{f}(\gamma)$ denotes in this paper what Hewitt and Ross call the coefficient operator, cf. [7, (34.3)(a)].)

Following [7] we denote by $\mathbb{S} = \mathbb{S}(\Gamma)$ the space consisting of functions $W$ on $\Gamma$ such that $W(\gamma)$ is a linear transformation on $\mathbb{H}_\gamma$ for each $\gamma \in \Gamma$.

**Definition 1.** An element $w \in \mathbb{S}$ is called a multiplier of $L^p(G)$ ($1 \leq p \leq \infty$) if for every $f \in L^p(G)$ the series $\sum_{\gamma \in \Gamma} d_\gamma \text{tr}(W(\gamma) \hat{f}(\gamma) D_\gamma(x))$, is the Fourier series of an element $T_wf$ of $L^p(G)$. The space of multipliers is denoted by $M_p(G)$.

We notice that the operators $T_wf$ of $L^p$ into $L^p$ are linear and (by the closed graph theorem) continuous. We can endow therefore $M_p(G)$ with a Banach space norm: the norm of $W \in M_p(G)$ is defined to be the norm of the

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corresponding operator $T_W$ on $L^p(G)$. It is easy to verify, for $p < \infty$, that
the operators $T_W$, with $W \in M_p$, are exactly the bounded linear operators on
$L^p(G)$ which commute by left translations.

**Definition 2.** A multiplier $W \in M_p(G)$ is said to vanish at infinity if
\[ \lim_{y \to \infty} \|W(y)\| = 0, \]
where the norm is that of $W(y)$ as an operator on $\mathbb{S}_\gamma$.

It is known that if $G$ is an Abelian group and $p \neq 2$, there exist multi-
pliers which vanish at infinity and are not the limit, in the norm of $M_p$, of
multipliers with finite support. For $p = 1$ or $p = \infty$ this is equivalent to the
classical result which asserts the existence of singular measures with Fourier-Stieltjes transform vanishing at infinity. For $1 < p < \infty$, $p \neq 2$, this was
proved in [4].

We remark that multipliers which are the limit in $M_p(G)$ of finitely sup-
ported ones are precisely those for which the corresponding operator on $L^p(G)$
is compact and that the elements of $\mathcal{S}(\Gamma)$ which vanish at infinity correspond
to compact operators on $L^2(G)$.

The purpose of the present paper is to extend the results described a-
bove, when $1 < p < \infty$ and $p \neq 2$, to a class of noncommutative compact
groups. We shall prove in fact the following:

**Theorem A.** Let $J$ be an infinite index set and let $G = \prod_{i \in J} G_i$, where
for each $i$, $G_i$ is a nontrivial compact group. Let $1 < p < \infty$ and $p \neq 2$. Then
there exists a multiplier $W \in M_p(G)$ which vanishes at infinity and is not
the limit, in the norm of $M_p$, of finitely supported elements of $\mathbb{S}$.

The proof of this theorem is based on two lemmas. The first lemma is
due to C. Fefferman and H. S. Shapiro [2, Theorem 1] and the second to
A. Bonami [1, pp. 374–375]. Both lemmas were stated and proved only for
commutative compact groups, but proofs can be easily translated into the
language of noncommutative groups, as will be indicated below.

**Lemma 1 (Fefferman and Shapiro).** Let $1 < p < \infty$, then there exists a
constant $\alpha = \alpha(p) > 0$ such that if $W \in M_p(G)$, and $W$ satisfies the condi-
tions: (i) $W(\gamma_0) = 0$, where $\gamma_0$ is the equivalence class of the trivial repre-
sentation, (ii) $\|W\|_{M_p} \leq \alpha(p)$; then the multiplier defined by $W'(\gamma_0) = I$ (the
identity operator), and $W'(\gamma) = W(\gamma)$ for $\gamma \neq \gamma_0$, has norm one.

The proof of this lemma is almost exactly the same as that which is
given in [2] for the corresponding result for commutative groups. Only two
remarks are needed. First of all the norm-decreasing inclusion $M_p(G) \subseteq M_2(G)$,
a well-known fact for $G$ commutative, is a consequence, for noncommutative
MULTIPLIERS VANISHING AT INFINITY

G, of recent results of C. Herz [6, Theorem C]. Second, the proof of Theorem 1 in [2] makes use of the fact that \( M_p = M_q \) for commutative \( G \), when \( 1/p + 1/q = 1 \). This equality is not known to be true for noncommutative \( G \), but we can use the known fact that \( M_p = M_q' \), where \( M_q' \) is the space of “right” multipliers of \( L^q \), that is the space of \( W \in \mathbb{C} \) such that for \( f \in L^q \), the Fourier-series \( \sum_{\gamma \in \Gamma} d^g f(x) W(y) D_\gamma(x) \), represents a function in \( L^q [3] \). With these two remarks in mind the proof of Theorem 1 of [2] is easily reinterpreted to yield Lemma 1.

Before stating the second lemma we remark that if \( G = G_1 \times G_2 \) where \( G_1 \) and \( G_2 \) are compact groups with dual objects \( \Gamma_1 \) and \( \Gamma_2 \), respectively, then the dual object \( \Gamma \) of \( G \) can be written as \( \Gamma = \Gamma_1 \times \Gamma_2 \), in the sense that if \( \gamma \in \Gamma \), there exists a unique pair \( (\gamma_1, \gamma_2) \in \Gamma_1 \times \Gamma_2 \), such that any representative \( D_\gamma \) of \( \gamma \) is unitarily equivalent to the tensor product \( D_{\gamma_1} \otimes D_{\gamma_2} \) of a representative \( D_{\gamma_1} \) of \( \gamma_1 \) and a representative \( D_{\gamma_2} \) of \( \gamma_2 \). We shall then write \( \gamma = \gamma_1 \times \gamma_2 \) [7, Theorem 27.4.3].

Lemma 2 (A. Bonami). Let \( G_1 \) and \( G_2 \) be compact groups, with dual objects \( \Gamma_1 \) and \( \Gamma_2 \) and let \( G = G_1 \times G_2 \). Let \( W_1 \) and \( W_2 \) be elements of \( M_p(G_1) \) and \( M_p(G_2) \), respectively, and suppose that \( \|W_1\|_p = \|W_2\|_p = 1 \). (As before \( \gamma_0 \) denotes the class containing the trivial representation.) Then the element \( W \in \mathbb{C}(\Gamma_1 \times \Gamma_2) \) defined by \( W(\gamma) = W_1(\gamma_1) \otimes W_2(\gamma_2) \), if \( \gamma = \gamma_1 \times \gamma_2 \), is an element of \( M_p(G) \) and \( \|W\|_p \leq 1 \).

Again, the proof of this lemma is exactly as in the commutative case [1, Lemma 1, p. 375].

Proof of Theorem A. If \( G = \Pi_{i \in J} G_i \) where \( G_i \) are nontrivial groups and \( J \) is an infinite set, we may assume, without loss of generality, that each \( G_i \) is infinite (if not divide \( J \) into infinitely many infinite subsets and group together the factors). We may also assume for simplicity that \( J \) is countable and in fact that \( J = \{1, 2, \ldots, l\} \). For each \( i \) we know that since \( p \neq 2 \) the norm-decreasing inclusion \( M_p(G_i) \subseteq M_2(G_i) \) is strict [5, Theorem 6]. This implies that the norm of \( M_p(G_i) \) is not equivalent to that of \( M_2(G_i) \) because \( M_p \) is not closed in \( M_2 \). Therefore we can find a finitely supported \( W_i \in M_p(G_i) \) such that \( W_i(\gamma_0) = 0 \), \( \|W_i\|_M \leq 1/i \), \( \|W_i\|_M = \alpha(p) \), where \( \alpha(p) \) is the constant appearing in the statement of Lemma 1. Let \( W_i' \) be the multiplier satisfying \( W_i'(\gamma_0) = 1 \), \( W_i'(\gamma) = W_i(\gamma) \) for \( \gamma \neq \gamma_0 \), whose norm is one by Lemma 1.

Applying inductively Lemma 2 we can construct elements \( W^{(n)}(\gamma) \in M_p(G^{(n)}) \) where \( G^{(n)} = \Pi_{i=1}^n G_i \), such that if \( \Gamma^{(n)} \) is the dual object of \( G^{(n)} \), and \( \gamma \in \Gamma^{(n)} \), \( \gamma = \gamma_1 \times \cdots \times \gamma_n \), then \( W^{(n)}(\gamma) = W_1'(\gamma_1) \otimes \cdots \otimes W_n'(\gamma_n) \) and \( \|W^{(n)}\|_M \leq 1 \).
Obviously we may consider $W^{(n)}$ as an element of $M_p(G)$ with the same norm, by defining $W^{(n)}(\gamma) = 0$, if $\gamma \not\in \Gamma_1 \times \cdots \times \Gamma_n$, where $\Gamma_i$ is the dual object of $G_i$. Finally let $W$ be a weak* limit of $W^{(n)}$ (we consider $M_p$ as the dual space of $A_p$ [3]). Then $W(\gamma) = W^{(n)}(\gamma)$ if $\gamma \in \Gamma_1 \times \cdots \times \Gamma_n$. We must show that $W$ vanishes at infinity and is not the limit in the norm of $M_p$ of finitely supported multipliers. Let $\epsilon > 0$ be given and let $1/n < \epsilon$. Denote by $K_n$ the finite set $K_n = \{\gamma_1 \times \cdots \times \gamma_n; \gamma_i \in \text{supp } \gamma_i \subseteq \Gamma_i\}$. Let $\gamma \notin K_n$ and suppose $W(\gamma) \neq 0$.

Let $\gamma = \gamma_1 \times \cdots \times \gamma_m$; with $\gamma_i \in \Gamma_i$ and $\gamma_m \neq \gamma_0$. Since $\gamma \notin K_n$ and $W(\gamma) \neq 0$, then $m > n$. Now $W(\gamma) = W_1'(\gamma_1) \otimes \cdots \otimes W_m'(\gamma_m)$. Therefore $\|W(\gamma)\| \leq \|W_1'(\gamma_1)\| \cdots \|W_m'(\gamma_m)\| \leq \|W_m'(\gamma_m)\|$, but since $\gamma_m \neq \gamma_0$, $W_m'(\gamma_m) = W_m'(\gamma_m)$; therefore,

$$\|W(\gamma)\| \leq \|W_m'(\gamma_m)\| \leq \sup_{\gamma \in \Gamma_m} \|W_m'(\gamma)\| = \|W_m\|_{M_2} < 1/n.$$ 

We have proved that if $\gamma \in K_n$, $\|W(\gamma)\| < 1/n$ and hence $W$ vanishes at infinity. On the other hand $W$ cannot be the limit in the norm of $M_p$ of finitely supported multipliers, for suppose $\tilde{W}$ is a finitely supported multiplier satisfying $\|\tilde{W} - W\|_{M_p} < \alpha(p)/4$. Since the support of $\tilde{W}$ is finite, for some $m$, $\Gamma_m \cap \text{supp } W \subseteq \{\gamma_0\}$.

Now $\|W_m\|_{M_p} > \alpha(p)/2$ so there exists a trigonometric polynomial $f$ with $\|f\|_p = 1$, supp $\hat{f} \subseteq \Gamma_m$ such that $\|T_{W_m}f\|_p > \alpha(p)/2$.

Let $E \in \mathcal{S}(\Gamma)$ denote the characteristic function of $\gamma_0$. Then $E \in M_p(G)$, $\|E\|_{M_p} = 1$, and $E + W_m = W_m'$. Also $W|\Gamma_m = W_m'$ and $W(\gamma_0) = 1$. Thus one verifies that

$$(T_W - T_{\tilde{W}})f = T_E(T_W - T_{\tilde{W}})f + T_{W_m}f.$$ 

Therefore

$$\frac{\alpha(p)}{4} > \|W - \tilde{W}\|_{M_p} \geq \|(T_W - T_{\tilde{W}})f\|_p$$

$$\geq \|T_{W_m}f\|_p - \|T_E(T_W - T_{\tilde{W}})f\|_p$$

$$> \alpha(p)/2 - \alpha(p)/4 = \alpha(p)/4,$$

a contradiction. This completes the proof of the Theorem.

REFERENCES


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