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PROPERTIES OF NEARLY-COMPACT SPACES

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ABSTRACT. A general product theorem for nearly-compact spaces and locally nearly-compact spaces is given along with other relating properties.

1. Introduction. A topological space (X, T) is said to be *nearly-compact* [7] if every open cover of X has a finite subcollection, the interiors of the closures of which cover X; a topological space (X, T) is said to be *locally nearly-compact* [2] if each point has an open neighborhood whose closure is a nearly-compact subset of X. It has been shown [7] that the product of two nearly-compact spaces is nearly-compact. The primary purpose of this paper is to prove a general product theorem for nearly-compact spaces. Using this result, a theorem pertaining to the product of locally nearly-compact spaces is obtained. Throughout, $\mathrm{cl}(A)$ will denote the closure of a set A and $\mathrm{Int}(A)$ will denote the interior of a set A.

2. Product of nearly-compact spaces. In a topological space (X, T) a set A is called *regular-open* if $A = \mathrm{Int}(\mathrm{cl}(A))$ and *regular-closed* if $A = \mathrm{cl}(\mathrm{Int}(A))$ [3, p. 92]. Since the intersection of two regular-open sets is regular-open, the regular-open sets form a base for a smaller *semiregular* topology T_* on X, called the semiregularization of T [1, p. 138]. We call a closed (open) set A of a space (X, T) *star-closed* (*star-open*) if and only if A is closed (open), respectively, in (X, T_*). Thus it follows that each star-closed set is the intersection of a collection of regular-closed sets and each star-open set is the union of a collection of regular-open sets.

Remark 2.1. Let U be a regular-open set in the space $S = \Pi_\alpha\{Y_\alpha: Y_\alpha$ has topology T_α, $\alpha \in \Delta\}$ (S has the usual product topology denoted by T) containing the point $\{y_\alpha^0\} \in \Pi_\alpha Y_\alpha$. By noting the behavior of the closure and interior operators on basic open sets, it follows that there exists a finite collection of regular-open sets $\{U_{\alpha_i}\}_{i=1}^n$ (U_{α_i} regular-open in Y_{α_i}) and a

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regular-open set $G = \Pi_{\alpha \neq \alpha_i} Y_\alpha \times U_{\alpha_1} \times U_{\alpha_2} \times \cdots \times U_{\alpha_n}$ containing $\{y_\alpha^0\}$ such that $\{y_\alpha^0\} \in G \subset U$. Now if we let $S_* = \Pi_\alpha \{Y_\alpha: Y_\alpha$ has topology T_{α_*}, $\alpha \in \Delta\}$ and denote its topology by T_0. It follows from the preceding discussion that the semiregular topology, T_*, associated with the space (S, T) is equal to the topology T_0.

Remark 2.2. A topological space (X, T) is *nearly-compact* if and only if each regular-open cover admits a finite subcover [7]. It was noted in [2, Theorem 4.1] that a space (X, T) is nearly-compact if and only if (X, T_*) is compact. Using this result along with Remark 2.1, we next give a general product theorem for nearly-compact spaces.

Theorem 2.1. *Let* $\{(Y_\alpha, T_\alpha): \alpha \in \Delta\}$ *be any family of spaces. Then* $\Pi_\alpha Y_\alpha$ *is nearly-compact if and only if each* Y_α *is nearly-compact.*

Proof. Assume that the $\Pi_\alpha Y_\alpha$ is nearly-compact. Since each projection $P_\beta: \Pi_\alpha Y_\alpha \to Y_\beta$ is a continuous open surjection, [7, Theorem 3.1] shows that each Y_α is nearly-compact.

Conversely, assume that each (Y_α, T_α) is nearly-compact and let $S = \Pi_\alpha \{Y_\alpha: Y_\alpha$ has topology T_α and $\alpha \in \Delta\}$ and $S_* = \Pi_\alpha \{Y_\alpha: Y_\alpha$ has topology T_{α_*} and $\alpha \in \Delta\}$. Now let $\{U_\beta: U_\beta$ regular-open in S, $\beta \in I\}$ be a regular-open cover of the space S. Using Remark 2.1 and Remark 2.2 it follows that $\{U_\beta\}$ is a star-open cover in the compact space S_*. Consequently, S is nearly-compact. This completes the proof.

Definition 2.1. A topological space (X, T) is called locally nearly-compact if each point has an open neighborhood whose closure is a nearly-compact subset of X [2].

[2, Theorem 4.5] shows that if (X, T) is a locally nearly-compact Hausdorff space, then (X, T_*) is locally compact. We next show that the converse to [2, Theorem 4.5] is also true.

Lemma 2.1. *Let* (X, T) *be a topological space. Then* (X, T) *is a locally nearly-compact Hausdorff space if and only if* (X, T_*) *is a locally-compact Hausdorff space.*

Proof. Only the sufficiency requires proof. Assume that (X, T_*) is a locally-compact Hausdorff space and let $x \in X$. Clearly (X, T) is Hausdorff. There exists a star-open set V in (X, T_*) containing x such that $\mathrm{cl}_*(V)$ (closure in (X, T_*)) is a compact subset in (X, T_*). Since $\mathrm{cl}_*(V) = \mathrm{cl}(V)$, $\mathrm{cl}(V)$ is a nearly-compact subset in (X, T). Therefore (X, T) is locally nearly-compact.

Theorem 2.2. *Let $\{(Y_\alpha, T_\alpha): \alpha \in \Delta\}$ be a family of Hausdorff spaces. Then $\Pi_\alpha Y_\alpha$ is locally nearly-compact if and only if all the Y_α are locally nearly-compact and at most finitely many are not nearly-compact.*

Proof. The proof is a standard, well-known argument after noting Remark 2.1 and Lemma 2.1.

3. **General properties.** A topological space (X, T) is said to be *completely Hausdorff* [1, p. 146] if each pair of distinct points of X has disjoint closed neighborhoods. A Hausdorff space (X, T) is said to be *H-closed* (called absolutely closed in [1]) if every open covering of X contains a finite subfamily whose closures cover X [6]. It follows from [7, Corollary 2.2] that a Hausdorff space X is nearly-compact if and only if it is an H-closed completely Hausdorff space.

Remark 3.1. We define a subset $A \subset X$ to be an H-closed subset if for every collection $\{U_\alpha: U_\alpha \text{ open in } X, \alpha \in \Delta\}$ such that $A \subset \bigcup_\alpha U_\alpha$, there exists a finite subcollection $\{U_{\alpha_i}\}_{i=1}^n$, such that $A \subset \bigcup_{i=1}^n \mathrm{cl}(U_{\alpha_i})$. Since points in a Hausdorff space X can be separated by disjoint regular-open sets [1, p. 138], it follows that H-closed subsets are star-closed in X.

It is well known that the intersection of a descending family of continua in a compact Hausdorff space is a continuum [4, p. 43]. We next give an example to show that in general, a similar result need not hold in nearly-compact Hausdorff spaces.

Example 3.1. Let $I^2 = [0, 1] \times [0, 1]$ have as a subbasis the usual open sets in I^2 along with the complements of the set $A \times \{0\}$, where A runs through the set of all subsets of $[0, 1]$. It follows from [1, Exercise 23b, p. 147] that I^2 is a nearly-compact Hausdorff space, whose semiregular topology, T_*, is the usual cartesian product topology of I^2. For each $n = 1, 2, 3, \cdots$, let $F_n = \{(x, y) \in I^2 : y \leq 1/n\}$. Then $\{F_n\}_{n=1}^\infty$ is a descending family of nearly-compact connected subsets in I^2 such that $K = \bigcap_n F_n = [0, 1]$. It follows that K, as a subspace of I^2, is disconnected and not nearly-compact.

Remark 3.2. Even though the set K in Example 3.1 is not a nearly-compact subspace in I^2, it is a nearly-compact subset since it is star-closed. Also we note that K is not a regular-closed set in I^2 because $\mathrm{Int}\, K = \emptyset$. With this in consideration we give our next theorem.

Theorem 3.1. *Let (X, T) be a nearly-compact Hausdorff space and let ω_β be an initial ordinal. Suppose that $\{F_\alpha: \alpha < \omega_\beta\}$ is a descending family ($\alpha < \gamma$ implies $F_\alpha \supset F_\gamma$) of nonempty star-closed subsets in X. Then*

(a) $K = \bigcap_\alpha F_\alpha$ *will be a nonempty nearly-compact subset of* X;

(b) *if each* F_α *is connected in* (X, T) *and* $K = \bigcap_\alpha F_\alpha$ *is regular-closed, then* K *will be connected in* (X, T).

Proof. First, we observe, as we have already noted, (X, T) is a nearly-compact Hausdorff space if and only if (X, T_*) is a compact Hausdorff space. Since each F_α is star-closed in (X, T), it follows that each F_α is a compact subset in (X, T_*). Therefore $\bigcap_\alpha F_\alpha = K$ is a nonempty compact subset in (X, T_*) which implies that K is a nonempty nearly-compact subset in (X, T).

Now assume that each F_α is connected in (X, T) and $\bigcap_\alpha F_\alpha = K$ is regular-closed in (X, T). Clearly each F_α is connected in (X, T_*). Therefore, [4, Lemma 2.8, p. 43] gives $K = \bigcap_\alpha F_\alpha$ compact and connected in (X, T_*). Now suppose that K is not connected in (X, T). Then there exist two disjoint nonempty regular-closed sets C_1 and C_2 in the subspace K such that $K = C_1 \cup C_2$. Since K is regular-closed in (X, T) and C_i $(i = 1, 2)$ is regular-closed in K, it follows that C_i $(i = 1, 2)$ is regular-closed in (X, T) and consequently, closed in (X, T_*). Therefore, K is not connected in (X, T_*) which is a contradiction. We conclude that K is connected in (X, T). This completes the proof.

Remark 3.3. A function is *almost-continuous* if the inverse images of regular-open sets are open and a function is *almost-open* if the images of regular-open sets are open [8].

We say a map $f: X \to Y$ is star-closed if the images of star-closed sets are closed.

Theorem 3.2. *Let* $f: (X, T_0) \to (Y, T)$ *be an almost-continuous map of a nearly-compact space* X *into a Hausdorff space* Y. *Then the images of star-closed subsets in* X *will be* H*-closed subsets in* Y. *Moreover,* f *will be a star-closed map.*

Proof. Let A be a star-closed subset in X. Since X is nearly-compact, A is an H-closed subset in X. Now since $f: (X, T_0) \to (Y, T)$ is almost-continuous if and only if $f: (X, T_0) \to (Y, T_*)$ is continuous, it follows that $f(H)$ is an H-closed subset in (Y, T) and consequently closed. Therefore, f is a star-closed map. This completes the proof.

[5, Theorem 4] shows that when $f: X \to Y$ is an almost-continuous surjection of a connected space X onto a space Y, then Y will be connected. However, it was noted in [5] that almost-continuous maps do not preserve connected sets in general.

Our next theorem gives conditions when connected sets are preserved under almost-continuous maps.

Theorem 3.3. *Let $f: X \to Y$ be an almost-continuous almost-open map of a nearly-compact space X into a Hausdorff space Y. Then the image of each regular-closed connected set in X will be a regular-closed connected set in Y.*

Proof. Let H be a connected regular-closed set in X. We first show that $f(H) = F$ is regular-closed in Y. Theorem 3.2 assures us that F is closed. Suppose $\mathrm{cl}(\mathrm{Int}(F)) \subsetneq F$. Then there exists a $y \in F$ such that $y \notin \mathrm{cl}(\mathrm{Int}(F))$. Since $y \in f(H) = F$ there exists an $x \in H$ such that $f(x) = y$. Now $Y - \mathrm{cl}(\mathrm{Int}(F))$ is a regular-open set containing $f(x)$; therefore, there exists an open set W containing x such that $f(W) \subset Y - \mathrm{cl}(\mathrm{Int}(F))$. Hence, $f(W) \cap \mathrm{cl}(\mathrm{Int}(F)) = \emptyset$. Since $x \in H = \mathrm{cl}(\mathrm{Int}(H))$, $W \cap \mathrm{Int}(H) \neq \emptyset$. Therefore,

$$\emptyset \neq f[W \cap \mathrm{Int}(H)] \subset f(W) \cap f(\mathrm{Int}(H))$$
$$\subset f(W) \cap \mathrm{Int}\, f(H) \subset f(W) \cap \mathrm{cl}(\mathrm{Int}(F)).$$

Thus $f(W) \cap \mathrm{cl}(\mathrm{Int}(F)) \neq \emptyset$ which is a contradiction. We conclude that $f(H) = \mathrm{cl}(\mathrm{Int}(f(H)))$ is a regular-closed set in Y. Now assume that $f(H)$ is not connected in Y. Then there exist two nonempty, disjoint, regular-closed sets C_1 and C_2 in the subspace $f(H)$ such that $f(H) = C_1 \cup C_2$. Since $f(H)$ is regular-closed in Y, it follows that C_i $(i = 1, 2)$ is regular-closed in Y. Now the almost-continuity of f gives disjoint closed sets $(f^{-1}(C_1)) \cap H$ and $(f^{-1}(C_2)) \cap H$ in X such that $H = [(f^{-1}(C_1)) \cap H] \cup [(f^{-1}(C_2)) \cap H]$. Therefore H is disconnected, which is a contradiction. Hence, we conclude that $f(H)$ is a regular-closed connected subset in Y.

Theorem 3.4. *Let $f: (X, T) \to (Y, T_0)$ be a continuous map of a nearly-compact space X into a regular space Y. Then $f(X)$ will be a compact subset in Y.*

Proof. Since Y is regular, $f: (X, T) \to (Y, T_0)$ is continuous if and only if $f: (X, T_*) \to (Y, T_0)$ is continuous [6, Theorem 2.15]. Now Remark 2.2 gives (X, T_*) compact. Consequently, it follows that $f(X)$ is a compact subset in Y. This completes the proof.

Our final theorem is an application of the notion of locally nearly-compactness.

Theorem 3.5. *Let $f(X, T) \to (Y, T_0)$ be a continuous star-closed map of a locally nearly-compact Hausdorff space X into a regular space Y. If f has nearly-compact point inverses, then $f(X)$ will be a locally compact Hausdorff subspace of Y.*

Proof. We first observe that $f: (X, T) \to (Y, T_0)$ is a star-closed continuous map if and only if $f: (X, T_*) \to (Y, T_0)$ is a closed continuous map.

Also we note that $f^{-1}(y)$ is a nearly-compact subset in (X, T) if and only if $f^{-1}(y)$ is a compact subset in (X, T_*). Consequently, it follows from the hypothesis that $f: (X, T_*) \to (Y, T_0)$ is a proper map [1, p. 101]. Now by Lemma 2.1 we have that (X, T_*) is locally compact. Therefore $f: (X, T_*) \to (Y, T_0)$ is a proper mapping of a locally-compact Hausdorff space (X, T_*) into a space Y. Hence, the result follows from [1, Corollary 2, p. 100] and the corollary following [1, Proposition 9, p. 105].

Remark 3.4. If the space (Y, T_0) in Theorem 3.5 is given to be compact (resp. locally-compact), it does not imply the corresponding space (X, T) is compact (resp. locally-compact). For let R be the reals, and T the topology on R having the open intervals and the set $A = \{x \in R: x \text{ is rational and } 1/3 < x < 2/3\}$ as a subbasis. Give the set $X = [0, 1]$ the subspace topology of (R, T) and topologize the set $Y = [0, 1]$ with the usual (Euclidean) subspace topology of R. It follows that the space $X = [0, 1]$ is a nearly-compact (and consequently a locally nearly-compact) Hausdorff space which is not locally-compact. Let $1: X \to Y$ be the identity map of X onto Y. The identity map 1 satisfies the hypothesis of Theorem 3.5, but the space X is not locally-compact.

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