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THE HARDY CLASS OF A FUNCTION WITH SLOWLY-GROWING AREA

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ABSTRACT. In this paper we show that if f is analytic on the open unit disk and if the area of $[\{|z| \leq R\} \cap$ image of $f]$ grows sufficiently slowly as a function of R, then f belongs to the Hardy class H^p for all p satisfying $0 < p < +\infty$.

Let Δ denote the open unit disk $\{|z| < 1\}$. It has been shown recently by H. Alexander, B. A. Taylor and J. L. Ullman [1, Theorem 1, p. 335] that

(1) *if f is analytic on Δ and the area of $f(\Delta)$ is finite, then f belongs to the Hardy class H^2; and*

(2) *if in addition $f(0) = 0$, then $\frac{1}{2}\|f\|_2^2 \leq$ Area of $f(\Delta)$.*

In this note we prove a theorem which strengthens (1) above:

Theorem. *Let f be analytic on Δ. For $R > 0$, let*

$$A(R) = \text{Area of } [\{|z| \leq R\} \cap f(\Delta)].$$

If $A(R)(R^{-2} \log R) \to 0$ as $R \to +\infty$, then $f \in H^p$ for all p satisfying $0 < p < +\infty$.

We remark that while weakening the hypothesis of (1) by replacing the condition "$A(R)$ bounded" by the condition "$A(R)(R^{-2} \log R) \to 0$ as $R \to +\infty$," we have been able to strengthen the conclusion and get $f \in H^p$ for *all* $0 < p < +\infty$. This new hypothesis is almost best possible since the inequality $A(R)R^{-2} \leq \pi$ holds for any complex-valued function f.

We remark further that the proof will make no use of the fact that the domain of f is the unit disk, and hence the Theorem remains valid if Δ is replaced by any region.

Proof. Without loss of generality, we assume that $f(0) = 0$ and that f is unbounded. For $t > 0$, we define

$\theta(t)$ = Angular Lebesgue measure of $[\{|z| = t\} \cap f(\Delta)]$,

$\alpha(t)$ = Angular Lebesgue measure of longest subarc of $[\{|z| = t\} \cap f(\Delta)]$,

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and

$$\chi(t) = 0 \quad \text{if } \{|z| = t\} \subset f(\Delta),$$
$$= 1 \quad \text{if } \{|z| = t\} \not\subset f(\Delta).$$

For $R > 1$, let

$$B(R) = \frac{\pi}{\log R} \int_1^R \frac{\chi(t)}{t\alpha(t)}\, dt.$$

Theorem 3.1 of [2] states that $f \in H^p$ for all p satisfying $0 < p < \liminf_{R \to +\infty} B(R)$, and thus our theorem will follow if we can show that $B(R) \to +\infty$ as $R \to +\infty$. From the definitions of θ and α we get the inequalities

$$A(R) = \int_0^R t\theta(t)\, dt \geq \int_0^R t\alpha(t)\, dt$$
$$\geq \int_{[0,R]\cap\{\alpha(t)=2\pi\}} t\alpha(t)\, dt = 2\pi \int_{[0,R]\cap\{\alpha(t)=2\pi\}} t\, dt.$$

Let $m(R)$ = Lebesgue measure of $[0, R] \cap \{\alpha(t) = 2\pi\}$. Then, since $g(t) = t$ is an increasing function of t, we get

$$A(R) \geq 2\pi \int_0^{m(R)} t\, dt = \pi[m(R)]^2.$$

Multiplying both sides of this inequality by $R^{-2} \log R$ and recalling the original assumption on $A(R)$, we conclude that $R^{-1}m(R) \to 0$ as $R \to +\infty$.

Now $[R - m(R)] \leq \int_0^R \chi(t)\, dt$, and hence

$$[R - 1 - m(R)]^2 \leq \left(\int_1^R \chi(t)\, dt\right)^2$$

if $R - 1 - m(R) \geq 0$. An application of the Schwarz inequality yields

$$\left(\int_1^R \chi(t)\, dt\right)^2 \leq \left(\int_1^R t\alpha(t)\, dt\right)\left(\int_1^R \frac{\chi(t)}{t\alpha(t)}\, dt\right) \leq A(R)(1/\pi)B(R) \log R.$$

Since $R - 1 - m(R) \geq 0$ for large R, we get the inequality

$$\pi[R - 1 - m(R)]^2/A(R) \log R \leq B(R).$$

Using the fact that $R^{-1}m(R) \to 0$ and $A(R)R^{-2} \log R \to 0$ as $R \to +\infty$, we conclude that $B(R) \to +\infty$ as $R \to +\infty$, which completes the proof.

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