REDUCING SUBSPACES OF CONTRACTIONS WITH NO ISOMETRIC PART

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ABSTRACT. Let $T$ be a contraction on a Hilbert space $H$ and suppose that there is no nonzero vector $f$ in $H$ such that $\|T^n f\| = \|f\|$ for every $n = 1, 2, \cdots$. In this paper, the reducing subspaces of $T$ are characterized in terms of the range of $1 - T^*T$. As a corollary, it is shown that $T$ is irreducible if $1 - T^*T$ has 1-dimensional range. In particular, if $U$ is the simple unilateral shift, then the restriction of $U^*$ to any invariant subspace for $U^*$ is irreducible.

Let $T$ be a contraction on a Hilbert space $H$. We recall that a (closed) subspace $M$ of $H$ reduces $T$ if $M$ is invariant under both $T$ and $T^*$. If the only subspaces that reduce $T$ are $\{0\}$ and $H$ itself, then $T$ is said to be irreducible. The contraction $T$ has no isometric part if there is no nonzero vector $f$ in $H$ such that $\|T^n f\| = \|f\|$ for every $n = 1, 2, \cdots$. The structure of the reducing subspaces for the adjoint of a unilateral shift is well known [2, Lemma 3.2, p. 724], [3, Theorem 1, p. 105]. In the present paper, the structure is obtained for arbitrary contractions with no isometric part.

If $E$ is a subset of $H$, then $\overline{VE}$ will denote the closed span of $E$. For subspaces $M$ and $N$ of $H$ such that $N \subset M$, $M \ominus N$ will denote the orthogonal complement of $N$ in $M$.

Theorem. Let $T$ be a contraction on a Hilbert space $H$ and suppose that $T$ has no isometric part. Let $K$ be the closure of the range of $1 - T^*T$. A subspace $M$ of $H$ reduces $T$ if and only if $M = \overline{\bigcup \{T^n S \cap N \cap \overline{S}^* : f \in S, n \geq 0\}}$ for some unique subspace $S$ of $K$ which is invariant under $(1 - T^*T)T^m T^n$ for every $m, n = 0, 1, 2, \cdots$. In this case

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Proof. Suppose that $M$ reduces $T$ and let $S$ be the closure of $(1 - T^*T)M$ in $H$. Clearly $\sqrt{\{T^n/f: f \in S, n \geq 0\}} \subset M$, and if $g$ is in $M \ominus \sqrt{\{T^n/f: f \in S, n \geq 0\}}$, then

$$(g, T^n(1 - T^*T)T^n g) = 0$$

for every $n = 0, 1, 2, \cdots$. Hence $\|T^{n+1}g\| = \|T^ng\|$ for every $n = 0, 1, 2, \cdots$. Since $T$ has no isometric part, $g = 0$. Therefore

$$M = \sqrt{\{T^n/f: f \in S, n \geq 0\}}.$$  

Since $M$ reduces $T$, we have that $K \ominus S$ is the closure in $H$ of $(1 - T^*T)(H \ominus M)$ and therefore $H \ominus M = \sqrt{\{T^n/f: f \in K \ominus S, n \geq 0\}}$.

Conversely, suppose that $M = \sqrt{\{T^n/f: f \in S', n \geq 0\}}$ where $S'$ is a subspace of $K$ which is invariant under $(1 - T^*T)T^mT^n$ for every $m, n = 0, 1, 2, \cdots$. Let

$$N = \{g \in H: (1 - T^*T)T^mg \in S', \forall m = 0, 1, 2, \cdots\}.$$  

Clearly $N \supset M$. Let $g$ belong to $N$. Since $(1 - T^*T)T^ng$ is in $S'$, we have that $T^*(1 - T^*T)T^n g$ is in $M$ for every $n = 0, 1, 2, \cdots$. It follows as above that $N = M$ and hence that $M$ reduces $T$.

Let $S$ be the closure of $(1 - T^*T)M$ in $H$. Clearly $S \subset S'$ since $(1 - T^*T)T^n f$ is in $S'$ for every $f$ in $S'$ and for every $n = 0, 1, 2, \cdots$. As above, $H \ominus M = \sqrt{\{T^n/f: f \in K \ominus S, n \geq 0\}}$. It follows that $S' \ominus S$ is contained in both $M$ and $H \ominus M$, and consequently $S' = S$.

Corollary 1. If $T$ is a contraction with no isometric part and $1 - T^*T$ has 1-dimensional range, then $T$ is irreducible.

Proof. In the Theorem, let $K$ be 1-dimensional. Since the only subspaces $S$ of $K$ are therefore $\{0\}$ and $K$ itself, it follows that the only subspaces $M$ of $H$ that reduce $T$ are $\{0\}$ and $H$ itself.

A basic result is that the restriction of the simple unilateral shift $U$ to any invariant subspace for $U$ is irreducible [3]. Corollary 1 immediately implies

Corollary 2. The restriction of $U^*$ to any invariant subspace for $U^*$ is irreducible.

Remark. The above theorem was obtained by proving special cases with the use of the canonical model of de Branges and Rovnyak [1].
REFERENCES


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