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ON WEAKLY CONTINUOUS MAPPINGS

TAKASHI NOIRI

ABSTRACT. P. E. Long and D. A. Carnahan studied several properties of almost continuous mappings in the sense of Singal (denoted by a.c.S.) in Proc. Amer. Math. Soc. 38(1973), 413–418. In this note, it is shown that "a.c.S." in some theorems of the above paper can be replaced by "weakly continuous".

1. **Introduction.** In 1966 T. Husain [1] introduced the concept of almost continuous mappings and investigated some of their properties. On the other hand, in 1968 M. K. Singal and Asha Rani Singal [5] have also introduced the concept, similarly called almost continuous mappings, which is in fact different from that in the sense of Husain. Quite recently, P. E. Long and D. A. Carnahan [3] have studied similarities and dissimilarities between these two kinds of almost continuity. The concept of weak continuity, due to N. Levine [2], is weaker than that of almost continuity in the sense of Singal and Singal [5, Example 2.3]. The purpose of the present note is to point out that "almost continuous mapping in the sense of Singal and Singal" in some theorems of [3] may be replaced by "weakly continuous mapping".

2. **Definitions and notations.** Let A be a subset of a topological space. The closure of A and the interior of A are denoted by $\mathrm{Cl}(A)$ and $\mathrm{Int}(A)$ respectively. Throughout the note, X and Y denote topological spaces, and by $f: X \to Y$ we denote a mapping f of a space X into a space Y.

Definition 1. A mapping $f: X \to Y$ is said to be *almost continuous* in the sense of Singal and Singal (briefly a.c.S.) if for each point $x \in X$ and each open set V in Y containing $f(x)$, there exists an open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Int}(\mathrm{Cl}(V))$ [5].

Definition 2. A mapping $f: X \to Y$ is said to be *weakly continuous* (briefly w.c.) if for each point $x \in X$ and each open set $V \subset Y$ containing $f(x)$, there exists an open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Cl}(V)$ [2].

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Remark 1. We have *continuous* $\Rightarrow$ *a.c.S.* $\Rightarrow$ *w.c.*, but none of these implications is reversible in general, while the three conditions are equivalent if the range is a regular space [2], [5].

Definition 3. A mapping $f: X \to Y$ is said to be *almost continuous* in the sense of Husain (briefly a.c.H.) if for each point $x \in X$ and each open set $V \subset Y$ containing $f(x)$, $\mathrm{Cl}(f^{-1}(V))$ is a neighborhood of x in X [1].

Remark 2. Example 2.1 in [5] and Example 1 in [4] show that Definitions 2 and 3 are completely independent of each other.

3. **Weakly continuous mappings.** The following lemma will be very useful.

Lemma 1 (Levine [2]). *A mapping $f: X \to Y$ is w.c. if and only if for each open set $V \subset Y$, $f^{-1}(V) \subset \mathrm{Int}(f^{-1}(\mathrm{Cl}(V)))$.*

Similarly to a.c.S. and a.c.H. mappings, w.c. mappings have the following property.

Theorem 1. *Let $f: X \to Y$ be a mapping and $g: X \to X \times Y$ be the graph mapping of f, given by $g(x) = (x, f(x))$ for every point $x \in X$. Then $g: X \to X \times Y$ is w.c. if and only if $f: X \to Y$ is w.c.*

Proof. *Necessity.* Suppose g is w.c. Let $x \in X$ and $V \subset Y$ be any open set containing $f(x)$. Then $X \times V$ is an open set in $X \times Y$ containing $g(x)$. Since g is w.c., there exists an open set $U \subset X$ containing x such that $g(U) \subset \mathrm{Cl}(X \times V) = X \times \mathrm{Cl}(V)$. Since g is the graph mapping of f, we have $f(U) \subset \mathrm{Cl}(V)$. This shows that f is w.c.

Sufficiency. Suppose f is w.c. Let $x \in X$ and W be any open set in $X \times Y$ containing $g(x)$. Then there exist open sets $R \subset X$ and $V \subset Y$ such that $g(x) = (x, f(x)) \in R \times V \subset W$. Since f is w.c., there exists an open set $U \subset X$ containing x such that $U \subset R$ and $f(U) \subset \mathrm{Cl}(V)$. Therefore, we have $g(U) \subset R \times \mathrm{Cl}(V) \subset \mathrm{Cl}(R \times V) \subset \mathrm{Cl}(W)$. This shows that g is w.c.

By a *w.c. retraction* we mean a w.c. mapping $f: X \to A$ where $A \subset X$ and $f|A$ is the identity mapping on A. The following theorem is a slight improvement of Theorem 3 in [3].

Theorem 2. *Let $A \subset X$ and $f: X \to A$ be a w.c. retraction of X onto A. If X is Hausdorff, then A is a closed subset of X.*

Proof. Suppose A is not closed. Then, there exists a point $x \in \mathrm{Cl}(A) - A$. Since f is a w.c. retraction, we have $f(x) \neq x$. Since X is Hausdorff, there exist disjoint open sets U and V such that $x \in U$ and $f(x) \in V$. Thus we obtain $U \cap \mathrm{Cl}(V) = \emptyset$. Now let W be any open set in X containing x.

Then $U \cap W$ is an open set containing x and hence $(U \cap W) \cap A \neq \emptyset$ because $x \in \mathrm{Cl}(A)$. Therefore, there exists a point $y \in U \cap W \cap A$. Since $y \in A$, $f(y) = y \in U$ and hence $f(y) \notin \mathrm{Cl}(V)$. This shows that $f(W) \not\subset \mathrm{Cl}(V)$. This contradicts the hypothesis that f is w.c. Thus A is a closed set in X.

It is shown in Theorem 4 of [3] that connectedness is invariant under a.c.S. surjections. The following theorem shows that "a.c.S." in the above theorem can be replaced by "w.c."

Theorem 3. *If X is a connected space and $f: X \to Y$ is a w.c. surjection, then Y is connected.*

Proof. Suppose Y is not connected. Then, there exist nonempty open sets V_1 and V_2 in Y such that $V_1 \cap V_2 = \emptyset$ and $V_1 \cup V_2 = Y$. Hence we have $f^{-1}(V_1) \cap f^{-1}(V_2) = \emptyset$ and $f^{-1}(V_1) \cup f^{-1}(V_2) = X$. Since f is surjective, $f^{-1}(V_j) \neq \emptyset$ for $j = 1, 2$. By Lemma 1, we have $f^{-1}(V_j) \subset \mathrm{Int}\,(f^{-1}(\mathrm{Cl}(V_j)))$ because f is w.c. Since V_j is open and also closed, we have $f^{-1}(V_j) \subset \mathrm{Int}\,(f^{-1}(V_j))$. Hence $f^{-1}(V_j)$ is open in X for $j = 1, 2$. This implies that X is not connected. This is contrary to the hypothesis that X is connected. Therefore, Y is connected.

4. Weakly continuous mappings and a.c.H. mappings. The following theorem shows that "open a.c.S." in Theorem 7 of [3] can be replaced by "w.c."

Theorem 4. *If $f: X \to Y$ is w.c., then $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for each open set $V \subset Y$.*

Proof. Suppose there exists a point $x \in \mathrm{Cl}(f^{-1}(V)) - f^{-1}(\mathrm{Cl}(V))$. Then $f(x) \notin \mathrm{Cl}(V)$. Hence there exists an open set W containing $f(x)$ such that $W \cap V = \emptyset$. Since V is open, we have $V \cap \mathrm{Cl}(W) = \emptyset$. Since f is w.c., there exists an open set $U \subset X$ containing x such that $f(U) \subset \mathrm{Cl}(W)$. Thus we obtain $f(U) \cap V = \emptyset$. On the other hand, since $x \in \mathrm{Cl}(f^{-1}(V))$, we have $U \cap f^{-1}(V) \neq \emptyset$ and hence $f(U) \cap V \neq \emptyset$. We have a contradiction. Thus we have $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$.

Theorem 5. *If $f: X \to Y$ is a.c.H. and $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ for each open set $V \subset Y$, then f is w.c.*

Proof. For any point $x \in X$ and any open set $V \subset Y$ containing $f(x)$, by the hypothesis we have $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$. Since f is a.c.H., there exists an open set $U \subset X$ such that $x \in U \subset \mathrm{Cl}(f^{-1}(V))$. Thus we have $f(U) \subset \mathrm{Cl}(V)$. This implies that f is w.c.

Corollary 1. *An a.c.H. mapping $f: X \to Y$ is w.c. if and only if* $\mathrm{Cl}(f^{-1}(V)) \subset f^{-1}(\mathrm{Cl}(V))$ *for every open set* $V \subset Y$.

Proof. This follows immediately from Theorems 4 and 5.

5. Weakly continuous mappings and Urysohn spaces. A space X is called a *Urysohn* space if for every pair of distinct points x and y in X, there exist open sets U and V in X such that $x \in U$, $y \in V$ and $\mathrm{Cl}(U) \cap \mathrm{Cl}(V) = \emptyset$.

Theorem 6. *If Y is a Urysohn space and $f: X \to Y$ is a w.c. injection, then X is Hausdorff.*

Proof. For any distinct points $x_1, x_2 \in X$, we have $f(x_1) \neq f(x_2)$ because f is injective. Since Y is Urysohn, there exist open sets V_1 and V_2 in Y such that $f(x_1) \in V_1$, $f(x_2) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Hence we have $\mathrm{Int}\,(f^{-1}(\mathrm{Cl}(V_1))) \cap \mathrm{Int}\,(f^{-1}(\mathrm{Cl}(V_2))) = \emptyset$. Since f is w.c., by Lemma 1, we have $x_j \in f^{-1}(V_j) \subset \mathrm{Int}\,(f^{-1}(\mathrm{Cl}(V_j)))$ for $j = 1, 2$. This implies that X is Hausdorff.

Theorem 7. *If f_1 and f_2 are two w.c. mappings of a space X into a Urysohn space Y, then $\{x \in X \mid f_1(x) = f_2(x)\}$ is closed in X.*

Proof. By A we denote the set $\{x \in X \mid f_1(x) = f_2(x)\}$. If $x \in X - A$, then we have $f_1(x) \neq f_2(x)$. Since Y is Urysohn, there exist two open sets V_1 and V_2 in Y such that $f_1(x) \in V_1$, $f_2(x) \in V_2$ and $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Since f_j is w.c., by Lemma 1, we have $x \in f_j^{-1}(V_j) \subset \mathrm{Int}\,(f_j^{-1}(\mathrm{Cl}(V_j)))$ for $j = 1, 2$. Let us put $U = \mathrm{Int}\,(f_1^{-1}(\mathrm{Cl}(V_1))) \cap \mathrm{Int}\,(f_2^{-1}(\mathrm{Cl}(V_2)))$; then U is an open set in X such that $x \in U \subset X - A$ because $\mathrm{Cl}(V_1) \cap \mathrm{Cl}(V_2) = \emptyset$. Thus $X - A$ is open in X. Consequently, A is closed in X.

The following corollary is a generalization of the well-known principle of extension of identities.

Corollary 2. *Let f_1 and f_2 be w.c. mappings of a space X into a Urysohn space Y. If B is dense in X and $f_1 = f_2$ on B, then $f_1 = f_2$.*

Proof. This is an immediate consequence of Theorem 7.

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DEPARTMENT OF MATHEMATICS, MIYAKONOJO TECHNICAL COLLEGE, MIYAKONOJO, MIYAZAKI-KEN, JAPAN

Current address: Department of Mathematics, Yatsushiro Technical College, Yatsushiro-shi, Kumamoto-ken, Japan