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# ON SUMS OF HANKEL OPERATORS[1]

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ABSTRACT. Necessary and sufficient conditions are derived for the sum of two Hankel operators of closed range to have closed range. As a corollary we determine when two left invariant subspaces of $H^2$ have positive angle.

In this note we investigate the range of the sum of two Hankel operators. Let us denote by $\mathbf{T}$ the unit circle $\{\lambda: |\lambda| = 1\}$ and by $D$ the open unit disc $\{\lambda: |\lambda| < 1\}$. $H^2$ will denote the usual scalar Hardy space [5] identified also as the subspace of $L^2(\mathbf{T})$ of functions having vanishing negative Fourier coefficients. We denote by $P_{H^2}$ the orthogonal projection of $L^2(\mathbf{T})$ onto $H^2$. Let $J$ be the unitary map in $L^2(\mathbf{T})$ defined by $(Jf)(e^{it}) = f(e^{-it})$. Let $\phi \in H^\infty$; the Hankel operator $H_\phi$ corresponding to $\phi$ is the bounded operator in $H^2$ defined by

$$H_\phi f = P_{H^2}\phi J f \quad \text{for all } f \in H^2. \tag{1}$$

It is clear from the definition that $(\text{Range } H_\phi)^-$ is a left invariant subspace of $H^2$. In [3] the following theorem of D. N. Clark has been proved.

**Theorem A.** *The Hankel operator* $H$ *has closed range if and only if* $\phi$ *has a representation* $\phi = qg$ *with* $q$ *an inner function and* $g \in \overline{H}_0^\infty$, *i.e.* $g$ *is bounded conjugate analytic vanishing at* $\infty$, *and such that there exists a* $\delta > 0$ *for which*

$$|G(z)| + |\tilde{q}(z)| \geq \delta \quad \textit{for all } z \in D, \tag{2}$$

*where* $\tilde{q}(z) = (q(\bar{z}))^-$ *and* $G(e^{it}) = e^{-it}g(e^{-it})$.

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It follows in this case that Range $H_\phi = \{qH^2\}^\perp$.

In particular if $H_\phi$ has closed range then $\phi$ is noncyclic for the backward (left) shift [1].

Let us consider now two functions $\phi_i \in H^\infty$ which are noncyclic for the backward shift. Then $(\text{Range } H_{\phi_i})^-$ are proper left invariant subspaces and hence there exist inner functions $q_i$ for which $\{\text{Range } H_{\phi_i}\}^\perp = q_i H^2$.

**Theorem B.** (a) $(\text{Range } H_{\phi_1+\phi_2})^- = (\text{Range } (H_{\phi_1} + H_{\phi_2}))^- = \{q_1 q_2 H^2\}^\perp$ *if and only if* $q_1$, $q_2$ *have no common nontrivial inner factor.*

(b) *Suppose* Range $H_{\phi_i} = \{q_i H^2\}^\perp$, *then* Range $H_{\phi_1+\phi_2} = \{q_1 q_2 H^2\}^\perp$ *if and only if for some* $\delta > 0$ *and all* $z \in D$ *we have* $|q_1(z)| + |q_2(z)| \geq \delta$.

**Proof.** (a) $(\text{Range } H_\phi)^- = \{qH^2\}^\perp$ implies, since $\phi \in \{qH^2\}^\perp$, that $\phi = qg$ with $g \in \overline{H}_0^\infty$. Let $\tau_q$: $L^2(\mathrm{T}) \longrightarrow L^2(\mathrm{T})$ be the unitary map given by

$$(\tau_q f)(e^{it}) = e^{-it}\tilde{q}(e^{it}) f(e^{-it})$$

then $\tau(\{qH^2\}^\perp) = \{\tilde{q} H^2\}^\perp$ [2]. In particular $(\text{Range } H_\phi)^- = \{qH^2\}^\perp$ if and only if $\tau\phi$ is a cyclic vector for the restricted right shift in $\{\tilde{q} H^2\}^\perp$ and this occurs if and only if $\tau\phi$, $\tilde{q}$ have no common nontrivial inner factor [4]. But

$$(\tau\phi)(e^{it}) = e^{-it} q(e^{it}) q(e^{-it}) g(e^{-it}) = G(e^{it}).$$

So $(\text{Range } H_\phi)^- = \{qH^2\}^\perp$ is equivalent to $\phi = qg$ with $G$, $\tilde{q}$ having no common nontrivial inner factor.

Assume now $q_1$, $q_2$ to have no common nontrivial inner factor. Then $\phi_1 + \phi_2 = q_1 g_1 + q_2 g_2$, and on applying $\tau_{q_1 q_2}$ we get

$$G = \tau_{q_1 q_2}(\phi_1 + \phi_2) = \tilde{q}_2 G_1 + \tilde{q}_1 G_2$$

and

$$(\text{Range } (H_{\phi_1} + H_{\phi_2}))^- = (\text{Range } H_{\phi_1+\phi_2})^- = \{q_1 q_2 H^2\}^\perp$$

if and only if $G$, $\tilde{q}_1 \tilde{q}_2$ have no common nontrivial inner factor. Suppose they have a common nontrivial inner factor $\psi$. Then $\psi | G$ and $\psi | \tilde{q}_1 \tilde{q}_2$. Without loss of generality we may assume $\psi | \tilde{g}_1$. But then $\psi | \tilde{q}_2 G_1$ and since $\psi$ and $G_1$ have no common inner factor by the assumption $(\text{Range } H_\phi)^- = \{q_1 H^2\}^\perp$, then $\psi | \tilde{q}_2$ contrary to the assumption that $q_1$, $q_2$ have no common inner factor. Thus $(\text{Range } H_{\phi_1+\phi_2})^- = \{q_1q_2H^2\}^\perp$.

Conversely assume $(\text{Range } H_{\phi_1+\phi_2})^- = \{q_1q_2H^2\}^\perp$; then $\phi_1 + \phi_2 = q_1q_2g$ for some $g \in \overline{H}_0^\infty$ and $G$, $\tilde{q}_1\tilde{q}_2$ have no common inner factor. Suppose $q_1$, $q_2$ have a common inner factor $\psi$. Then $\tilde{\psi}|\tilde{q}_i$. Hence $\tilde{\psi}|(\tilde{q}_2G_1 + \tilde{q}_1G_2)$ that is $\tilde{\psi}|\tilde{G}$. But this contradicts the assumption that $(\text{Range } H_{\phi_1+\phi_2})^- = \{q_1q_2H^2\}^\perp$.

(b) Let us assume not that range $H_\phi = \{q_iH^2\}^\perp$, $i = 1, 2$; then there exists a $\delta > 0$ such that $|G_i(z)| + |q_i(z)| \geq \delta$ for all $z \in D$. Assume that for some $\delta_1 > 0$ we have $|q_1(z)| + |q_2(z)| \geq \delta_1$ for all $z \in D$. We will show that Range $H_{\phi_1+\phi_2} = \{q_1q_2H^2\}^\perp$. For this it suffices that $|G(z)| + |\tilde{q}_1\tilde{q}_2(z)| \geq \delta_2 > 0$ for all $z$ in $D$. If this condition is not satisfied there exists a sequence $\{z_n\}$ in $D$ for which $G(z_n) \to 0$ and $\tilde{q}_1(z_n)\tilde{q}_2(z_n) \to 0$. By passing to a subsequence we may assume without loss of generality that $\tilde{q}_1(z_n) \to 0$. Since $G(z) = \tilde{q}_1G_2 + \tilde{q}_2G_1$, it follows that $\tilde{q}_2(z_n)G_1(z_n) \to 0$. Now $G_1(z_n) \to 0$ is ruled out by $|G_1(z)| + |\tilde{q}_1(z)| \geq \delta$ whereas $\tilde{q}_2(z_n) \to 0$ is ruled out by $|q_1(z)| + |q_2(z)| \geq \delta_1$. So indeed Range $H_{\phi_1+\phi_2} = \{q_1q_2H^2\}^\perp$.

Conversely assume Range $H_{\phi_1+\phi_2} = \{q_1q_2H^2\}^\perp$. Then $G = \tilde{q}_1G_2 + \tilde{q}_2G_1$ and $|G(z)| + |\tilde{q}_1\tilde{q}_2(z)| \geq \delta$; i.e.

$$|\tilde{q}_1(z)G_2(z) + \tilde{q}_2(z)G_1(z)| + |\tilde{q}_1\tilde{q}_2(z)| \geq \delta > 0$$

for all $z$ in $D$. But since $G_i \in H^\infty$, this implies $|q_1(z)| + |q_2(z)| \geq \delta_1 > 0$ for all $z$ in $D$.

Given any proper left invariant subspace $K$ of $H^2$ then $K$ is the range of a Hankel operator in $H^2$. In fact by Beurling's theorem, $K = \{qH^2\}^\perp$ for some inner function $q$. Let $\phi(z) = (q(z) - q(0))/z$; then $\phi \in H^\infty \cap K$ and it is simple to check that, by Theorem A, Range $H_\phi = \{qH^2\}^\perp$. It is trivial that $\{q_1H^2\}^\perp \cap \{q_2H^2\}^\perp = \{0\}$ if and only if $q_1$, $q_2$ have no common nontrivial inner factor.

**Corollary 1.** *Let $q_1$, $q_2$ be inner functions; then $\{q_1H^2\}^\perp + \{q_2H^2\}^\perp = \{q_1q_2H^2\}^\perp$ if and only if there exists a $\delta > 0$ such that $|q_1(z)| + |q_2(z)| \geq \delta$ for all $z \in D$.*

Now it is well known [6, p. 243] that the sum of two subspaces $M_1$, $M_2$ of any Banach space, which satisfy $M_1 \cap M_2 = \{0\}$, is a closed subspace if and only if for some $d > 0$, $\inf\{\|x_1 - x_2\|: x_i \in M_i, \|x_i\| = 1\} \geq d$. In a Hilbert space this condition is equivalent to $\sup\{|(x_1, x_2)|: x_i \in M_i, \|x_i\| = 1\} < 1$

which can be interpreted geometrically as $M_1$, $M_2$ having a positive angle. Thus we get the following corollary.

**Corollary 2.** *The angle between* $\{q_1H^2\}^\perp$ *and* $\{q_2H^2\}^\perp$ *is positive if and only if for some* $\delta > 0$, $|q_1(z)| + |q_2(z)| \geq \delta$ *for all* $z \in D$.

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