ON SUBDIRECT PRODUCTS OF RINGS WITHOUT
SYMMETRIC DIVISORS OF ZERO

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ABSTRACT. A theorem of V. A. Andrunakievic and Ju. M. Rjabuhin
asserts that a ring $R$ is without nilpotent elements if and only if $R$
is a subdirect product of skew-domains. In this paper we prove that a
semiprime ring $R$ with involution is a subdirect product of rings with-
out symmetric divisors of zero if and only if $R$ is compressible for its
symmetric elements.

Introduction. Let $R$ be a ring with involution $*$, $S$ the set of all sym-
metric elements ($x = x^*$) of $R$, $T$ the set of all traces ($t + t^*$) of $R$, $N$ the
set of all norms ($tt^*$) of $R$, and $S_0 = T \cup N \subseteq S$. Given a symmetric ideal
$I = I^*$ of $R$, the factor ring $R/I$ will be equipped with the canonical invo-
lution $x + I \rightarrow x^* + I$. $R$ is said to be $*$-compressible if for any $x \in S_0$ and
$n$ equal to a power of 2, $ax^n b = 0$ implies $axb = 0$.

Andrunakievic and Rjabuhin have shown that a ring $R$ is without nil-
potents if and only if $R$ is a subdirect product of skew-domains [1]. Re-
lated to this, one would like to ask when $R$ has a subdirect representation
into rings without symmetric divisors of zero. We prove here that a semi-
prime ring $R$ is $*$-compressible if and only if $R$ is a subdirect product of
rings without symmetric divisors of zero (in the case of 2-torsion free).

Remark 1. Let $R = F_2$ be the ring of matrices over a field $F$. Assume
that for a given involution $*$, the symmetric divisors of zero are zero alone.
Then $*$ must be the symplectic involution (i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$). In fact
if $*$ were not the symplectic involution then, by a result of Jacobson
[7, Case A, p.311], $*$ must be of the following type:

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where $F$ has an involution $\lambda \to \bar{\lambda}$ and $\bar{r}_i = r_i$ ($i = 1, 2$) are fixed invertible symmetric elements of $F$. Now take $x = (0 \ 0 \ 1 \ 0)$; then $x^2 = 0$. Then

$$xx^* = \begin{pmatrix} 0 & 0 \\ 0 & r_1^{-1}r_2 \end{pmatrix} \neq 0 \quad \text{and} \quad x(xx^*) = 0.$$ 

This means $xx^*$ is a symmetric divisor of zero, a contradiction. This shows that $*$ must be the symplectic one.

**Remark 2.** C. Lanski has characterized a semiprime ring $R$ such that all its nonzero symmetric elements do not annihilate themselves in the case of 2-torsion free; namely, that $R$ has one of the following types: (i) a skew-domain, (ii) a subring of the direct sum of a skew-domain and its opposite with interchanging co-ordinate involution, or (iii) an order in $2 \times 2$ matrices over a field [9]. Montgomery and Herstein extended Lanski's characterization to the case where $R$ is any semiprime ring with involution such that all its nonzero traces do not annihilate themselves. In case (i), $R$ is, of course, *-compressible. In case (ii), $R$ is a subring of the direct sum of *-compressible rings, and, consequently, $R$ is again *-compressible. In case (iii), by using Remark 1 we get $S_0 \subseteq Z(F_2)$, the center of $F_2$, and so $R$ is *-compressible. This shows that any semiprime ring with involution such that all its nonzero traces do not annihilate themselves is a *-compressible ring.

We first prove several propositions.

**Proposition 1.** Let $R$ be a *-compressible ring.

1. If $s \in S_0$ with $s^n = 0$, then $s = 0$.
2. If $xx^* = 0$ then $x^*x = 0$.

**Proof.** (1) follows immediately from the definition of a *-compressible ring. As for (2), note that $(x^*x)^2 = 0$ and apply (1).

Recall that a ring $R$ is *-prime if for any symmetric ideals $A = A^*$ and $B = B^*$ of $R$, $AB = 0$ implies $A = 0$ or $B = 0$.

**Proposition 2.** Let $R$ be a *-prime ring. Then $R$ is *-compressible if and only if every nonzero symmetric element in $S_0$ is a nondivisor of zero.

**Proof.** "If" part. This is an immediate consequence of Remark 2. "Only if" part. First we show that $sd = 0$ with $s, d \in S_0$ implies $s = 0$ or $d = 0$. 

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Let \( x \in R \). We have
\[
s(sxd + dx^*s)^2d = s(sxsdxd + dx^*s^2xd + sx^2d^*s + dx^*sdx^*s)d = 0
\]
(because \( ds = sd = 0 \)).

By the hypothesis, \( s(sxd + dx^*s)d = 0 \). This means that \( s^2xd^2 = 0 \ \forall x \in R \).

Since \( R \) is \(*\)-prime, \( s^2 = 0 \) or \( d^2 = 0 \). Since \( R \) is \(*\)-compressible, \( s = 0 \) or \( d = 0 \). By Lanski's characterization, mentioned in Remark 2, \( R \) is without nonzero symmetric divisors of zero in \( S_0 \).

Recall that a subset \( M_0 \) of a ring \( R \) is an \( m \)-system if \( a, b \in M_0 \) imply \( ab \in M_0 \) for some \( x \in R \).

**Proposition 3.** Let \( R \) be a \(*\)-compressible ring. Let \( M_0 \) be an \( m \)-system excluding \((0)\). Then \((0)\) can be enlarged to a \(*\)-prime ideal \( P \) such that \( R/P \) is \(*\)-compressible and \( P \cap M_0 = \emptyset \).

**Proof.** Let \( M \) be an \( m \)-system containing \( M_0 \) and maximal with respect to the exclusion of the ideal \((0)\). As is well known, if \( J \) is the complement of \( M \), \( J = C(M) \), then \( J \) is a prime ideal of \( R \) excluding \( M_0 \). Assuming for the moment that \( J \) has the following property:

for any \( s \in S_0 \) and \( xs^n y \in J \) which imply \( xsy \in J \), where \( n \) is a power of \( 2 \), we take \( P = J \cap J^* \), and so have a \(*\)-prime ideal \( P \) such that \( R/P \) is \(*\)-compressible and \( P \cap M_0 = \emptyset \).

It remains to show that \( J \) has property (1). Define
\[
M_1 = M \cup \{xs^n y | xsy \in M; \ n = 1, 2, \ldots, 2^h, \ldots\},
\]
\[
M_2 = M_1 \cup \{x_1s_1^n y_1 | x_1s_1y_1 \in M_1; \ n = 1, 2, \ldots, 2^h, \ldots\},
\]
\[
M_{k+1} = M_k \cup \{x_ks_k^n y_k | x_ks_ky_k \in M_k; \ n = 1, 2, \ldots, 2^h, \ldots\}.
\]

Let \( M' = \bigcup_k M_k \). Of course, \( M' \supseteq M \). Now \( M' \) excludes \((0)\), for if \( M_k \) excludes \((0)\), \( M_{k+1} \) will exclude \((0)\)(since \( R \) is \(*\)-compressible) and, by construction, \( M \) excludes \((0)\). By induction on \( k \), all \( M_k \) exclude \((0)\) and consequently \( M' \) excludes \((0)\).

Next we show that \( M' \) is an \( m \)-system. In fact, it is enough to show that for any pair \( a, b \in M_k \) there is \( c \in R \) such that \( abc \in M' \) (for \( M \subseteq M_1 \subseteq M_2 \subseteq \cdots \)). The property holds for \( k = 1 \). Now suppose it is true for \( k \); let us show this for \( k + 1 \). Let \( a, b \in M_{k+1} \). We wish to find a \( c \in R \) such that \( abc \in M' \). There are three cases:
Case 1. Both $a, b \in M_k$. Here, $c$ follows from the induction step.

Case 2. Both $a, b \notin \{xs^n|xsy \in M_k\}$. Here $a = xs^n y, b = zd^m t$ for some $x, y, t$ and $z$ such that $xsy$ and $zd^m t \in M_k$. By the induction hypothesis, there exists $c \in R$ such that $(xsy)c(zd^m t) \in M_{k'}$, for some $k'$. Then $xs^n(yycz)zd^m t \in M_{k'+1}$ and $(xs^n yc)z^m t \in M_{k'+2}$. Hence $acb \in M'$.

Case 3. If $a \in M_k$ and $b \notin \{xs^n|xsy \in M_k\}$, then we have $a \in M_k$ and $b = xs^ny$ where $xsy \in M_k$. Thus there exists $c \in R$ such that $ac(xsy) \in M_{k'}$, for some $k'$. So $(acx)s^ny \in M_{k'+1}$. Hence $acb \in M'$.

This shows that $M'$ is an $m$-system containing $M$. By the above, $M'$ excludes $(0)$. By maximality, $M' = M$ and so $J = C(M)$ has property (1). The proof is now complete.

We are now in a position to prove the following key theorem.

Theorem 1. Let $R$ be *-compressible, and $\eta$ the prime radical of $R$. Then $\eta = \bigcap P' | P'$ is a *-prime ideal of $R$ and $R/P'$ has no nonzero symmetric divisors of zero in $S_0$.

Proof. If $P$ is a prime ideal of $R$, then $M = C(P)$ is an $m$-system excluding $(0)$. By Proposition 3, the ideal $(0)$ can be enlarged to a *-prime ideal $P'$ such that $R/P'$ is *-compressible and $P' \cap M = \emptyset$. Thus for such a *-prime ideal $P'$, we have $P' \subseteq C(M) = P$ and $R/P'$ has no nonzero symmetric divisors of zero in $S_0$ (Proposition 2). Hence $\eta = \bigcap P' | P'$ is a *-prime ideal of $R$ and $R/P'$ has no nonzero symmetric divisors of zero in $S_0$. Since $\eta$ is the least semiprime ideal of $R$, we have the equality; hence the proof is complete.

We can now derive our main theorem.

Theorem 2. Let $R$ be a semiprime ring with involution. Then $R$ is *-compressible if and only if $R$ is a subdirect product of rings without nonzero symmetric divisors of zero in $S_0$.

Proof. "Only if" part. It follows immediately from Theorem 1. "If" part. Let $R$ be a subdirect product of rings without symmetric divisors of zero in $S_0$. Let $x, y \in R$ and $s \in S_0$ with $xs^ny = 0$. We prove that $xsy = 0$.

In fact, in each factor $R_i$, we have $x_i s^ny_i = 0$. Since $R_i$ is semiprime and without symmetric divisors of zero in $S_0, R_i$ is *-compressible (Remark 2). Thus $x_i s_i y_i = 0$ for all indices $i$. Therefore $xsy = 0$.

Corollary 1. Let $R$ be a semiprime and *-compressible ring. Then $R$ is a subring of a direct product of rings which are either skew-domains or orders in $2 \times 2$ matrices over a field.
Corollary 2. A semiprime ring $R$ is $*$-compressible if and only if $x^n y = 0$ implies $x y = 0$ for any $s \in S_0$ and any integer $n > 1$.

Remark 3. If $R$ is 2-torsion free, then we observe that if $S_0$ does not contain nonzero divisors of zero, then $S$ itself does not contain nonzero divisors of zero. Also, if $M$ is an $m$-system excluding (0) then $M' = \{2^k x | x \in M, k = 1, 2, \ldots \}$ is also an $m$-system excluding (0). For any $*$-prime ideal $P$ with $P \cap M' = \emptyset$, $R/P$ is 2-torsion free (for if $2x \in P$ with $x \notin P$ then $2R(x, x^*) \subseteq P$; since $P$ is $*$-prime we have $2R \subseteq P$, a contradiction).

Combining Remark 3 with Proposition 2 and Theorem 1, we have

Theorem 3. Let $R$ be a 2-torsion free and semiprime ring. Then $R$ is $*$-compressible if and only if $R$ is a subdirect product of rings without nonzero symmetric divisors of zero.

From Theorem 2 we derive the following theorems which generalize [8], [12].

Theorem 4. Let $R$ be a semiprime and $*$-compressible ring. If $s_1 s_2 \cdots s_n = 0$ with $s_i \in S_0$, then for any permutation $i_1, i_2, \ldots, i_n$ in the $i$'s $s_{i_1} s_{i_2} \cdots s_{i_n} = 0$ (that is, the product of the $s_i$'s is zero in any order).

Proof. Let $(\lambda^{s_i})_{\lambda \in I}$ be the image of $s_i$ under a subdirect representation as in Theorem 2. If $s_1 s_2 \cdots s_n = 0$, then $\lambda^{s_1} \lambda^{s_2} \cdots \lambda^{s_n} = 0$ for all $\lambda \in I$. Using the regularity condition in each factor $\lambda R$ of $R$, there must be $\lambda^{s_i} = 0$. Thus $\lambda^{s_{i_1}} \lambda^{s_{i_2}} \cdots \lambda^{s_{i_n}} = 0$ for all $\lambda \in I$. Consequently $s_{i_1} s_{i_2} \cdots s_{i_n} = 0$.

Remark 4. Let $R$ be as in Theorem 4. By using Theorem 2 and Lanksi's theorem, one can see that if $s x d y t = 0$ with $s, d, t \in S_0$ and $x, y \in R$, then $s d t x y = 0$.

$R$ is said to be $*$-von Neumann regular ($*$-regular) if for any $s \in S_0$ there is $x \in R$ such that $a = axa$.

Theorem 5. Let $R$ be as in Theorem 4. Assume that $R/P$ is $*$-regular for every $*$-prime ideal $P$. Then $R$ is $*$-regular.

Proof. Let $a \in S_0$ and $E$ be the set of all elements of the form $(a - ax_1 a)(a - ax_2 a) \cdots (a - ax_n a);$

$x_1, x_2, \ldots, x_n$ running over $R$. Clearly $E$ is closed under multiplication. Then $E$ is an $m$-system. We claim that $0 \in E$. For if $0 \notin E$, then by Prop-
osition 3, the ideal \((0)\) can be enlarged to a \(*\)-prime ideal \(P\) such that \(R/P\) is \(*\)-compressible and \(P \cap E = \emptyset\). By the hypothesis, \(R/P\) is a \(*\)-regular ring and consequently \(a - aya \in P\) for a suitable \(y\). But \(a - aya \in E\), a contradiction. We must conclude that \(0 \in E\), that is, for some \(x_i \in R\),

\[
a(1 - x_1 a) a(1 - x_2 a) \cdots a(1 - x_n a) = 0.
\]

By Remark 4, we have

\[
a^n(1 - x_1 a) \cdots (1 - x_n a) = 0.
\]

Thus \(a(1 - x_1 a) \cdots (1 - x_n a) = 0\). Since the product \((1 - x_1 a) \cdots (1 - x_n a)\) has the form \(1 - za\), \(a(1 - za) = 0\), that is, \(a = aza\).

**Remark 5.** A long standing conjecture of I. Kaplansky [8] was that a ring \(R\) is von Neumann regular iff \(R\) is a semiprime ring such that each prime image of \(R\) is von Neumann regular. This conjecture was settled in the affirmative by I. N. Herstein and E. T. Wong separately in the case of rings without nilpotent elements [12] (see also Fisher and Snider [3]). Theorem 5, whose proof follows the pattern of Herstein-Kaplansky's proof (see [8]), is a generalization of Wong's result.

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