ON SUBDIRECT PRODUCTS OF RINGS WITHOUT
SYMMETRIC DIVISORS OF ZERO

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ABSTRACT. A theorem of V. A. Andrunakievich and Ju. M. Rjabuhin
asserts that a ring $R$ is without nilpotent elements if and only if $R$
is a subdirect product of skew-domains. In this paper we prove that a
semiprime ring $R$ with involution is a subdirect product of rings with-
out symmetric divisors of zero if and only if $R$ is compressible for its
symmetric elements.

Introduction. Let $R$ be a ring with involution $*$, $S$ the set of all sym-
metric elements ($x = x^*$) of $R$, $T$ the set of all traces ($t + t^*$) of $R$, $N$ the
set of all norms ($tt^*$) of $R$, and $S_0 = T \cup N \subseteq S$. Given a symmetric ideal
$I = I^*$ of $R$, the factor ring $R/I$ will be equipped with the canonical invo-
lution $x + I \rightarrow x^* + I$. $R$ is said to be $*$-compressible if for any $x \in S_0$ and
$n$ equal to a power of 2, $ax^n b = 0$ implies $axb = 0$.

Andrunakievich and Rjabuhin have shown that a ring $R$ is without nil-
potents if and only if $R$ is a subdirect product of skew-domains [1]. Re-
lated to this, one would like to ask when $R$ has a subdirect representation
into rings without symmetric divisors of zero. We prove here that a semi-
prime ring $R$ is $*$-compressible if and only if $R$ is a subdirect product of
rings without symmetric divisors of zero (in the case of 2-torsion free).

Remark 1. Let $R = F_2$ be the ring of matrices over a field $F$. Assume
that for a given involution $*$, the symmetric divisors of zero are zero alone.
Then $*$ must be the symplectic involution (i.e., $(a \ b)^* = (d \ -b)
(c \ a)$). In fact if $*$ were not the symplectic involution then, by a result of Jacobson
[7, Case A, p.311], $*$ must be of the following type:

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where $F$ has an involution $\lambda \mapsto \bar{\lambda}$ and $\bar{r}_i = r_i$ ($i = 1, 2$) are fixed invertible symmetric elements of $F$. Now take $x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$; then $x^2 = 0$. Then

$$xx^* = \begin{pmatrix} 0 & 0 \\ 0 & r_1^{-1}r_2 \end{pmatrix} \neq 0 \quad \text{and} \quad x(xx^*) = 0.$$ 

This means $xx^*$ is a symmetric divisor of zero, a contradiction. This shows that $\ast$ must be the symplectic one.

**Remark 2.** C. Lanski has characterized a semiprime ring $R$ such that all its nonzero symmetric elements do not annihilate themselves in the case of 2-torsion free; namely, that $R$ has one of the following types: (i) a skew-domain, (ii) a subring of the direct sum of a skew-domain and its opposite with interchanging co-ordinate involution, or (iii) an order in $2 \times 2$ matrices over a field [9]. Montgomery and Herstein extended Lanski’s characterization to the case where $R$ is any semiprime ring with involution such that all its nonzero traces do not annihilate themselves. In case (i), $R$ is, of course, $\ast$-compressible. In case (ii), $R$ is a subring of the direct sum of $\ast$-compressible rings, and, consequently, $R$ is again $\ast$-compressible. In case (iii), by using Remark 1 we get $S_0 \subseteq Z(F',)$, the center of $F'$, and so $R$ is $\ast$-compressible. This shows that any semiprime ring with involution such that all its nonzero traces do not annihilate themselves is a $\ast$-compressible ring.

We first prove several propositions.

**Proposition 1.** Let $R$ be a $\ast$-compressible ring.

1. If $s \in S_0$ with $s^n = 0$, then $s = 0$.
2. If $xx^* = 0$ then $x^*x = 0$.

**Proof.** (1) follows immediately from the definition of a $\ast$-compressible ring. As for (2), note that $(x^*x)^2 = 0$ and apply (1).

Recall that a ring $R$ is $\ast$-prime if for any symmetric ideals $A = A^\ast$ and $B = B^\ast$ of $R$, $AB = 0$ implies $A = 0$ or $B = 0$.

**Proposition 2.** Let $R$ be a $\ast$-prime ring. Then $R$ is $\ast$-compressible if and only if every nonzero symmetric element in $S_0$ is a nondivisor of zero.

**Proof.** "If" part. This is an immediate consequence of Remark 2. "Only if" part. First we show that $sd = 0$ with $s, d \in S_0$ implies $s = 0$ or $d = 0$. 

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Let $x \in R$. We have

$$s(sx + dx^*s)^2d = s(sxsdx + dx^*s^2x + sx^2s + dx^*sdx^*s)d = 0$$

(by the hypothesis, $s(sx + dx^*s)d = 0$. This means that $s^2x^2d = 0 \forall x \in R$.

Since $R$ is *-prime, $s^2 = 0$ or $a^2 = 0$. Since $R$ is *-compressible, $s = 0$ or $d = 0$. By Lanski's characterization, mentioned in Remark 2, $R$ is without nonzero symmetric divisors of zero in $S_0$.

Recall that a subset $M_0$ of a ring $R$ is an $m$-system if $a, b \in M_0$ imply $ab \in M_0$ for some $x \in R$.

**Proposition 3.** Let $R$ be a *-compressible ring. Let $M_0$ be an $m$-system excluding (0). Then (0) can be enlarged to a *-prime ideal $P$ such that $R/P$ is *-compressible and $P \cap M_0 = \emptyset$.

**Proof.** Let $M$ be an $m$-system containing $M_0$ and maximal with respect to the exclusion of the ideal (0). As is well known, if $J$ is the complement of $M, J = C(M)$, then $J$ is a prime ideal of $R$ excluding $M_0$. Assuming for the moment that $J$ has the following property:

for any $s \in S_0$ and $xs^n y \in J$ which imply $xsy \in J$, where $n$ is a (1) power of 2, we take $P = J \cap J^*$, and so have a *-prime ideal $P$ such that $R/P$ is *-compressible and $P \cap M_0 = \emptyset$.

It remains to show that $J$ has property (1). Define

$$M_1 = M \cup \{xs^n y | xsy \in M; \quad n = 1, 2, \ldots, 2^h, \ldots\},$$

$$M_2 = M_1 \cup \{x_1s^n y_1 | x_1s_1y_1 \in M_1; \quad n = 1, 2, \ldots, 2^h, \ldots\},$$

$$\ldots$$

$$M_{k+1} = M_k \cup \{x_ks^n k y_k | x_ks_ky_k \in M_k; \quad n = 1, 2, \ldots, 2^h, \ldots\}.$$
Case 1. Both \(a, b \in M_k\). Here, \(c\) follows from the induction step.

Case 2. Both \(a, b \in \{xs^n y | xsy \in M_k\}\). Here \(a = xs^n y, b = zd^m t\) for some \(x, y, t, z\) such that \(xsy, zdt \in M_k\). By the induction hypothesis, there exists \(c \in R\) such that \((xsy)c(zdt) \in M_{k'}\) for some \(k'\). Then \(xs^n(yczdt) \in M_{k'+1}\) and \((xs^n y cz)d^m t \in M_{k'+2}\). Hence \(acb \in M'\).

Case 3. If \(a \in M_k\) and \(b \in \{xs^n y | xsy \in M_k\}\), then we have \(a \in M_k\) and \(b = xs^n y\) where \(xsy \in M_k\). Thus there exists \(c \in R\) such that \(ac(xsy) \in M_{k'}\), for some \(k'\). So \((ac)s^n y \in M_{k'+1}\). Hence \(acb \in M'\).

This shows that \(M'\) is an \(m\)-system containing \(M\). By the above, \(M'\) excludes \((0)\). By maximality, \(M' = M\) and so \(J = C(M)\) has property (1). The proof is now complete.

We are now in a position to prove the following key theorem.

Theorem 1. Let \(R\) be \(*\)-compressible, and \(\eta\) the prime radical of \(R\). Then \(\eta = \cap \{P' | P'\} = \cap \{P' | P'\} = \cap \{P' | P'\}\) is a \(*\)-prime ideal of \(R\) and \(R/P'\) has no nonzero symmetric divisors of zero in \(S_0\).

Proof. If \(P\) is a prime ideal of \(R\), then \(M = C(P)\) is an \(m\)-system excluding \((0)\). By Proposition 3, the ideal \((0)\) can be enlarged to a \(*\)-prime ideal \(P'\) such that \(R/P'\) is \(*\)-compressible and \(P' \cap M = \emptyset\). Thus for such a \(*\)-prime ideal \(P'\), we have \(P' \subseteq C(M) = P\) and \(R/P'\) has no nonzero symmetric divisors of zero in \(S_0\) (Proposition 2). Hence \(\eta \geq \cap \{P' | P'\}\) is a \(*\)-prime ideal of \(R\) and \(R/P'\) has no nonzero symmetric divisors of zero in \(S_0\). Since \(\eta\) is the least semiprime ideal of \(R\), we have the equality; hence the proof is complete.

We can now derive our main theorem.

Theorem 2. Let \(R\) be a semiprime ring with involution. Then \(R\) is \(*\)-compressible if and only if \(R\) is a subdirect product of rings without nonzero symmetric divisors of zero in \(S_0\).

Proof. "Only if" part. It follows immediately from Theorem 1. "If" part. Let \(R\) be a subdirect product of rings without symmetric divisors of zero in \(S_0\). Let \(x, y \in R\) and \(s \in S_0\) with \(xs^n y = 0\). We prove that \(xsy = 0\). In fact, in each factor \(R_i\), we have \(x_i s^n y_i = 0\). Since \(R_i\) is semiprime and without symmetric divisors of zero in \(S_0\), \(R_i\) is \(*\)-compressible (Remark 2). Thus \(x_i s_i y_i = 0\) for all indices \(i\). Therefore \(xsy = 0\).

Corollary 1. Let \(R\) be a semiprime and \(*\)-compressible ring. Then \(R\) is a subring of a direct product of rings which are either skew-domains or orders in \(2 \times 2\) matrices over a field.
Corollary 2. A semiprime ring \( R \) is \(*\)-compressible if and only if \( x^n y = 0 \) implies \( x y = 0 \) for any \( s \in S_0 \) and any integer \( n > 1 \).

Remark 3. If \( R \) is 2-torsion free, then we observe that if \( S_0 \) does not contain nonzero divisors of zero, then \( S \) itself does not contain nonzero divisors of zero. Also, if \( M \) is an \( m \)-system excluding \( (0) \) then \( M' = \{ 2^k x | x \in M, k = 1, 2, \ldots \} \) is also an \( m \)-system excluding \( (0) \). For any \(*\)-prime ideal \( P \) with \( P \cap M' = \emptyset \), \( R/P \) is 2-torsion free (for if \( 2x \in P \) with \( x \notin P \) then \( 2R(x, x^*) \subseteq P \); since \( P \) is \(*\)-prime we have \( 2R \subseteq P \), a contradiction).

Combining Remark 3 with Proposition 2 and Theorem 1, we have

Theorem 3. Let \( R \) be a 2-torsion free and semiprime ring. Then \( R \) is \(*\)-compressible if and only if \( R \) is a subdirect product of rings without nonzero symmetric divisors of zero.

From Theorem 2 we derive the following theorems which generalize [8], [12].

Theorem 4. Let \( R \) be a semiprime and \(*\)-compressible ring. If \( s_1 s_2 \cdots s_n = 0 \) with \( s_i \in S_0 \), then for any permutation \( i_1, i_2, \ldots, i_n \) in the \( i \)'s
\[ s_{i_1} s_{i_2} \cdots s_{i_n} = 0 \] (that is, the product of the \( s_i \)'s is zero in any order).

Proof. Let \( \lambda^s \lambda^e \) be the image of \( s_i \) under a subdirect representation as in Theorem 2. If \( s_1 s_2 \cdots s_n = 0 \), then \( \lambda^s_1 \lambda^s_2 \cdots \lambda^s_n = 0 \) for all \( \lambda \in I \). Using the regularity condition in each factor \( \lambda R \) of \( R \), there must be \( \lambda s_i = 0 \). Thus \( \lambda s_1 \lambda s_2 \cdots \lambda s_n = 0 \) for all \( \lambda \in I \). Consequently \( s_1 s_2 \cdots s_n = 0 \).

Remark 4. Let \( R \) be as in Theorem 4. By using Theorem 2 and Lanksi's theorem, one can see that if \( sxdyt = 0 \) with \( s, d, t \in S_0 \) and \( x, y \in R \), then \( sdtxy = 0 \).

\( R \) is said to be \(*\)-von Neumann regular (\(*\)-regular) if for any \( s \in S_0 \) there is \( x \in R \) such that \( a = axa \).

Theorem 5. Let \( R \) be as in Theorem 4. Assume that \( R/P \) is \(*\)-regular for every \(*\)-prime ideal \( P \). Then \( R \) is \(*\)-regular.

Proof. Let \( a \in S_0 \) and \( E \) be the set of all elements of the form
\[ (a - ax_1 a)(a - ax_2 a) \cdots (a - ax_n a); \]
\( x_1, x_2, \ldots, x_n \) running over \( R \). Clearly \( E \) is closed under multiplication. Then \( E \) is an \( m \)-system. We claim that \( 0 \in E \). For if \( 0 \notin E \), then by Prop-
osition 3, the ideal (0) can be enlarged to a *-prime ideal \( P \) such that \( R/P \) is *-compressible and \( P \cap E = \emptyset \). By the hypothesis, \( R/P \) is a *-regular ring and consequently \( a - ay a \in P \) for a suitable \( y \). But \( a - ay a \in E \), a contradiction. We must conclude that \( 0 \in E \), that is, for some \( x_i \in R \),

\[
a(1 - x_1a) a(1 - x_2a) \cdots a(1 - x_na) = 0.
\]

By Remark 4, we have

\[
a^n(1 - x_1a) \cdots (1 - x_na) = 0.
\]

Thus \( a(1 - x_1a) \cdots (1 - x_na) = 0 \). Since the product \( (1 - x_1a) \cdots (1 - x_na) \) has the form \( 1 - za \), \( a(1 - za) = 0 \), that is, \( a = aza \).

Remark 5. A long standing conjecture of I. Kaplansky [8] was that a ring \( R \) is von Neumann regular if \( R \) is a semiprime ring such that each prime image of \( R \) is von Neumann regular. This conjecture was settled in the affirmative by I. N. Herstein and E. T. Wong separately in the case of rings without nilpotent elements [12] (see also Fisher and Snider [3]). Theorem 5, whose proof follows the pattern of Herstein-Kaplansky's proof (see [8]), is a generalization of Wong's result.

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