ON SUBDIRECT PRODUCTS OF RINGS WITHOUT
SYMMETRIC DIVISORS OF ZERO

TAO-CHENG YIT

ABSTRACT. A theorem of V. A. Andrunakievic and Ju. M. Rjabuhin
asserts that a ring $R$ is without nilpotent elements if and only if $R$
is a subdirect product of skew-domains. In this paper we prove that a
semiprime ring $R$ with involution is a subdirect product of rings with-
out symmetric divisors of zero if and only if $R$ is compressible for its
symmetric elements.

Introduction. Let $R$ be a ring with involution $*$, $S$ the set of all sym-
metric elements $(x = x^*)$ of $R$, $T$ the set of all traces $(t + t^*)$ of $R$, $N$ the
set of all norms $(tt^*)$ of $R$, and $S_0 = T \cup N \subseteq S$. Given a symmetric ideal
$I = I^*$ of $R$, the factor ring $R/I$ will be equipped with the canonical invo-
lution $x + I \rightarrow x^* + I$. $R$ is said to be $\ast$-compressible if for any $x \in S_0$ and
$n$ equal to a power of 2, $ax^n b = 0$ implies $axb = 0$.

Andrunakievic and Rjabuhin have shown that a ring $R$ is without nil-
potents if and only if $R$ is a subdirect product of skew-domains [1]. Re-
lated to this, one would like to ask when $R$ has a subdirect representation
into rings without symmetric divisors of zero. We prove here that a semi-
prime ring $R$ is $\ast$-compressible if and only if $R$ is a subdirect product of
rings without symmetric divisors of zero (in the case of 2-torsion free).

Remark 1. Let $R = F_2$ be the ring of matrices over a field $F$. Assume
that for a given involution $*$, the symmetric divisors of zero are zero alone.
Then $*$ must be the symplectic involution (i.e., $\begin{pmatrix} a & b \\ c & d \end{pmatrix}*$ = $\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$). In fact
if $*$ were not the symplectic involution then, by a result of Jacobson
[7, Case A, p.311], $*$ must be of the following type:
where $F$ has an involution $\lambda \rightarrow \bar{\lambda}$ and $\bar{r}_i = r_i$ ($i = 1, 2$) are fixed invertible symmetric elements of $F$. Now take $x = (0 \ 1; 1 \ 0)$; then $x^2 = 0$. Then

$$xx^* = \begin{pmatrix} 0 & 0 \\ 0 & r_1^{-1} r_2 \end{pmatrix} \neq 0 \quad \text{and} \quad x(xx^*) = 0.$$ 

This means $xx^*$ is a symmetric divisor of zero, a contradiction. This shows that $\ast$ must be the symplectic one.

**Remark 2.** C. Lanski has characterized a semiprime ring $R$ such that all its nonzero symmetric elements do not annihilate themselves in the case of 2-torsion free; namely, that $R$ has one of the following types: (i) a skew-domain, (ii) a subring of the direct sum of a skew-domain and its opposite with interchanging co-ordinate involution, or (iii) an order in $2 \times 2$ matrices over a field [9]. Montgomery and Herstein extended Lanski's characterization to the case where $R$ is any semiprime ring with involution such that all its nonzero traces do not annihilate themselves. In case (i), $R$ is, of course, $\ast$-compressible. In case (ii), $R$ is a subring of the direct sum of $\ast$-compressible rings, and, consequently, $R$ is again $\ast$-compressible. In case (iii), by using Remark 1 we get $S_0 \subseteq Z(F_2)$, the center of $F_2$, and so $R$ is $\ast$-compressible. This shows that any semiprime ring with involution such that all its nonzero traces do not annihilate themselves is a $\ast$-compressible ring.

We first prove several propositions.

**Proposition 1.** Let $R$ be a $\ast$-compressible ring.

1. If $s \in S_0$ with $s^n = 0$, then $s = 0$.
2. If $xx^* = 0$ then $x^*x = 0$.

**Proof.** (1) follows immediately from the definition of a $\ast$-compressible ring. As for (2), note that $(x^*x)^2 = 0$ and apply (1).

Recall that a ring $R$ is $\ast$-prime if for any symmetric ideals $A = A^*$ and $B = B^*$ of $R$, $AB = 0$ implies $A = 0$ or $B = 0$.

**Proposition 2.** Let $R$ be an $\ast$-prime ring. Then $R$ is $\ast$-compressible if and only if every nonzero symmetric element in $S_0$ is a nondivisor of zero.

**Proof.** ''If'' part. This is an immediate consequence of Remark 2.

''Only if'' part. First we show that $sd = 0$ with $s, d \in S_0$ implies $s = 0$ or $d = 0$. 

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Let $x \in R$. We have

$$s(sxd + dx^*s)^2d$$

$$= s(sxsdxd + dx^*s^2xd + sxsd^2x^*s + dx^*sdx^*s)d = 0$$

(because $ds = sd = 0$).

By the hypothesis, $s(sxd + dx^*s)d = 0$. This means that $s^2xd^2 = 0 \forall x \in R$. Since $R$ is *-prime, $s^2 = 0$ or $a^2 = 0$. Since $R$ is *-compressible, $s = 0$ or $d = 0$. By Lanski's characterization, mentioned in Remark 2, $R$ is without nonzero symmetric divisors of zero in $S_0$.

Recall that a subset $M_0$ of a ring $R$ is an $m$-system if $a, b \in M_0$ imply $axb \in M_0$ for some $x \in R$.

**Proposition 3.** Let $R$ be a *-compressible ring. Let $M_0$ be an $m$-system excluding (0). Then (0) can be enlarged to a *-prime ideal $P$ such that $R/P$ is *-compressible and $P \cap M_0 = \emptyset$.

**Proof.** Let $M$ be an $m$-system containing $M_0$ and maximal with respect to the exclusion of the ideal (0). As is well known, if $J$ is the complement of $M$, $J = C(M)$, then $J$ is a prime ideal of $R$ excluding $M_0$. Assuming for the moment that $J$ has the following property:

for any $s \in S_0$ and $xs^n y \in J$ which imply $xsy \in J$, where $n$ is a power of 2, we take $P = J \cap J^*$, and so have a *-prime ideal $P$ such that $R/P$ is *-compressible and $P \cap M_0 = \emptyset$.

It remains to show that $J$ has property (1). Define

$$M_1 = M \cup \{xs^n y | xsy \in M; \ n = 1, 2, \ldots, 2^h, \ldots\},$$

$$M_2 = M_1 \cup \{x_1s^n_1y_1 | x_1s_1y_1 \in M_1; \ n = 1, 2, \ldots, 2^h, \ldots\},$$

$$\ldots$$

$$M_{k+1} = M_k \cup \{x_ks^nky_k | x_ks_ky_k \in M_k; \ n = 1, 2, \ldots, 2^h, \ldots\}.$$ 

Let $M' = \bigcup_k M_k$. Of course, $M' \supset M$. Now $M'$ excludes (0), for if $M_k$ excludes (0), $M_{k+1}$ will exclude (0) (since $R$ is *-compressible) and, by construction, $M$ excludes (0). By induction on $k$, all $M_k$ exclude (0) and consequently $M'$ excludes (0).

Next we show that $M'$ is an $m$-system. In fact, it is enough to show that for any pair $a, b \in M_k$ there is $c \in R$ such that $acb \in M'$ (for $M \subseteq M_1 \subseteq M_2 \subseteq \cdots$). The property holds for $k = 1$. Now suppose it is true for $k$; let us show this for $k + 1$. Let $a, b \in M_{k+1}$. We wish to find a $c \in R$ such that $acb \in M'$. There are three cases:
Case 1. Both \( a, b \in M_k \). Here, \( c \) follows from the induction step.

Case 2. Both \( a, b \in \{xs^n y | xsy \in M_k\} \). Here \( a = xs^n y, b = zd^m t \) for some \( x, y, t \) and \( z \) such that \( xsy \) and \( zdt \in M_k \). By the induction hypothesis, there exists \( c \in R \) such that \( (xsy)c(zdt) \in M_{k'} \) for some \( k' \). Then \( xs^n(yzc)zdt) \in M_{k'+1} \) and \( (xs^n yz)c(zdt) \in M_{k'+2} \). Hence \( acb \in M' \).

Case 3. If \( a \in M_k \) and \( b \in \{xs^n y | xsy \in M_k\} \), then we have \( a \in M_k \) and \( b = xs^n y \) where \( xsy \in M_k \). Thus there exists \( c \in R \) such that \( ac(xsyt) \in M_{k'} \), for some \( k' \). So \( (acx)s^n y \in M_{k'+1} \). Hence \( acb \in M' \).

This shows that \( M' \) is an \( m \)-system containing \( M \). By the above, \( M' \) excludes \( (0) \). By maximality, \( M' = M \) and so \( J = C(M) \) has property (1). The proof is now complete.

We are now in a position to prove the following key theorem.

**Theorem 1.** Let \( R \) be \(*\)-compressible, and \( \eta \) the prime radical of \( R \). Then \( \eta = \bigcap (P' \mid P') \) is a \(*\)-prime ideal of \( R \) and \( R/P' \) has no nonzero symmetric divisors of zero in \( S_0 \).

**Proof.** If \( P \) is a prime ideal of \( R \), then \( M = C(P) \) is an \( m \)-system excluding \((0) \). By Proposition 3, the ideal \((0) \) can be enlarged to a \(*\)-prime ideal \( P' \) such that \( R/P' \) is \(*\)-compressible and \( P' \cap M = \emptyset \). Thus for such a \(*\)-prime ideal \( P' \), we have \( P' \subseteq C(M) = P \) and \( R/P' \) has no nonzero symmetric divisors of zero in \( S_0 \) (Proposition 2). Hence \( \eta \subseteq \bigcap (P' \mid P') \) is a \(*\)-prime ideal of \( R \) and \( R/P' \) has no nonzero symmetric divisors of zero in \( S_0 \). Since \( \eta \) is the least semiprime ideal of \( R \), we have the equality; hence the proof is complete.

We can now derive our main theorem.

**Theorem 2.** Let \( R \) be a semiprime ring with involution. Then \( R \) is \(*\)-compressible if and only if \( R \) is a subdirect product of rings without nonzero symmetric divisors of zero in \( S_0 \).

**Proof.** "Only if" part. It follows immediately from Theorem 1. "If" part. Let \( R \) be a subdirect product of rings without symmetric divisors of zero in \( S_0 \). Let \( x, y \in R \) and \( s \in S_0 \) with \( xs^n y = 0 \). We prove that \( xsy = 0 \). In fact, in each factor \( R_i \), we have \( x_i s_i^n y_i = 0 \). Since \( R_i \) is semiprime and without symmetric divisors of zero in \( S_0 \), \( R_i \) is \(*\)-compressible (Remark 2). Thus \( x_i s_i y_i = 0 \) for all indices \( i \). Therefore \( xsy = 0 \).

**Corollary 1.** Let \( R \) be a semiprime and \(*\)-compressible ring. Then \( R \) is a subring of a direct product of rings which are either skew-domains or orders in \( 2 \times 2 \) matrices over a field.
Corollary 2. A semiprime ring $R$ is $\ast$-compressible if and only if $xs^n y = 0$ implies $xsy = 0$ for any $s \in S_0$ and any integer $n > 1$.

Remark 3. If $R$ is 2-torsion free, then we observe that if $S_0$ does not contain nonzero divisors of zero, then $S$ itself does not contain nonzero divisors of zero. Also, if $M$ is an $m$-system excluding (0) then $M' = \{2^k x | x \in M, k = 1, 2, \ldots \}$ is also an $m$-system excluding (0). For any $\ast$-prime ideal $P$ with $P \cap M' = \emptyset$, $R/P$ is 2-torsion free (for if $2x \in P$ with $x \notin P$ then $2R(x, x^*) \subseteq P$; since $P$ is $\ast$-prime we have $2R \subseteq P$, a contradiction).

Combining Remark 3 with Proposition 2 and Theorem 1, we have

**Theorem 3.** Let $R$ be a 2-torsion free and semiprime ring. Then $R$ is $\ast$-compressible if and only if $R$ is a subdirect product of rings without nonzero symmetric divisors of zero.

From Theorem 2 we derive the following theorems which generalize [8], [12].

**Theorem 4.** Let $R$ be a semiprime and $\ast$-compressible ring. If $s_1 s_2 \cdots s_n = 0$ with $s_i \in S_0$, then for any permutation $i_1, i_2, \ldots, i_n$ in the $i$'s $s_{i_1} s_{i_2} \cdots s_{i_n} = 0$ (that is, the product of the $s_i$'s is zero in any order).

**Proof.** Let $(\lambda^s_i)_{\lambda \in I}$ be the image of $s_i$ under a subdirect representation as in Theorem 2. If $s_1 s_2 \cdots s_n = 0$, then $\lambda^s_1 \lambda^s_2 \cdots \lambda^s_n = 0$ for all $\lambda \in I$. Using the regularity condition in each factor $\lambda R$ of $R$, there must be $\lambda^s_i = 0$. Thus $\lambda^s_{i_1} \lambda^s_{i_2} \cdots \lambda^s_n = 0$ for all $\lambda \in I$. Consequently $s_{i_1} s_{i_2} \cdots s_{i_n} = 0$.

**Remark 4.** Let $R$ be as in Theorem 4. By using Theorem 2 and Lanksi's theorem, one can see that if $sxdyt = 0$ with $s, d, t \in S_0$ and $x, y \in R$, then $sdtxy = 0$.

$R$ is said to be $\ast$-von Neumann regular ($\ast$-regular) if for any $s \in S_0$ there is $x \in R$ such that $a = axa$.

**Theorem 5.** Let $R$ be as in Theorem 4. Assume that $R/P$ is $\ast$-regular for every $\ast$-prime ideal $P$. Then $R$ is $\ast$-regular.

**Proof.** Let $a \in S_0$ and $E$ be the set of all elements of the form $(a - ax_1 a)(a - ax_2 a) \cdots (a - ax_n a)$; $x_1, x_2, \ldots, x_n$ running over $R$. Clearly $E$ is closed under multiplication. Then $E$ is an $m$-system. We claim that $0 \in E$. For if $0 \notin E$, then by Prop-
osition 3, the ideal \( (0) \) can be enlarged to a \(*\)-prime ideal \( P \) such that \( R/P \) is \(*\)-compressible and \( P \cap E = \emptyset \). By the hypothesis, \( R/P \) is a \(*\)-regular ring and consequently \( a - aya \in P \) for a suitable \( y \). But \( a - aya \in E \), a contradiction. We must conclude that \( 0 \in E \), that is, for some \( x_i \in R \),

\[
a(1 - x_1a)\cdots (1 - x_na) = 0.
\]

By Remark 4, we have

\[
a^n(1 - x_1a)\cdots (1 - x_na) = 0.
\]

Thus \( a(1 - x_1a)\cdots (1 - x_na) = 0 \). Since the product \( (1 - x_1a)\cdots (1 - x_na) \) has the form \( 1 - za \), \( a(1 - za) = 0 \), that is, \( a = aza \).

Remark 5. A long standing conjecture of I. Kaplansky [8] was that a ring \( R \) is von Neumann regular iff \( R \) is a semiprime ring such that each prime image of \( R \) is von Neumann regular. This conjecture was settled in the affirmative by I. N. Herstein and E. T. Wong separately in the case of rings without nilpotent elements [12] (see also Fisher and Snider [3]). Theorem 5, whose proof follows the pattern of Herstein-Kaplansky's proof (see [8]), is a generalization of Wong's result.

Acknowledgements. This paper is a part of the author's doctoral dissertation at Carleton University. The author gratefully acknowledges Professor Maurice Chacron for his advice and assistance.

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DEPARTMENT OF MATHEMATICS, CARLETON UNIVERSITY, OTTAWA, ONTARIO, CANADA