EVERY DIRECTION A JULIA DIRECTION

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ABSTRACT. Let \( f(z) = \exp(\cosh z) \). If \( N \) is any \( \epsilon \)-neighborhood of any ray through the origin with slope \( m \neq 0, \infty \) then \( f^{-1}(w) \cap N \) is infinite if \( w \neq 0 \).

Let \( J[f] \) denote the set of Julia directions of the entire function \( f \). That is, \( \theta \in J[f] \) if, in every sector \( \alpha < \arg z < \beta \) such that \( \alpha < \beta \), \( f \) assumes every complex value, with at most one exception, infinitely often. Using infinite products Julia [2] has constructed an entire function for which every direction is a Julia direction. The example which follows is more elementary.

The closed annulus \( \{1/n \leq |z| \leq n\} \) will be denoted \( A_n \) for \( n = 1, 2, \ldots \). If \( a, b, \delta > 0 \) the closed rectangle \( \{x + iy: -a < x < a; b - \delta < y < b + \delta\} \) will be denoted by \( R_\delta(a, b) \).

Theorem. If \( f(z) = \exp(\cosh z) \) then \( J[f] = \mathbb{R} \).

Proof. The relations \( f(\overline{z}) = (f(z))^\overline{\theta} \) and \( f(z) = f(-z) \) imply that if \( \theta \in J[f] \) then \( -\theta, \theta \pm \pi \in J[f] \). Consequently we can finish the proof by showing that \( (0, \pi/2) \subset J[f] \) because the set of Julia directions is clearly always closed.

Now suppose that \( \theta, \alpha, \beta \) are given and that they satisfy \( 0 < \theta < \pi/2 \) and \( \alpha < \theta < \beta \). They will be fixed for the rest of the proof.

Let \( S = \{x + iy: mx - \epsilon < y < mx + \epsilon\} \) where \( m = \tan \theta \) and \( 0 < \epsilon < \pi \).

The function \( z \rightarrow \frac{1}{2} \exp(z) \) maps the strip \( S \) onto a ribbon which starts at the origin, wraps around it infinitely often, and spirals out to infinity. This ribbon has \( y_- \) for its inside boundary and \( y_+ \) for its outside boundary where

\[
y_\pm(y) = \frac{1}{2} \exp[(y \pm \epsilon)/m + iy]
\]

for all \( y \). The width of the ribbon \( |y_+(y) - y_-(y)| \), which is measured along a ray from the origin, is an unbounded increasing function of \( y \). The ribbon does not overlap itself because \( |y_-(y + 2n)| - |y_+(y)| > 0 \) provided \( \epsilon < \pi \).

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Let $P_n$ denote the open parallelogram which is the intersection of the strip $\{x + iy : 2m < y < 2m + n\}$ with the strip $S$. Note that for all $n$ larger than some $n_0$ the sector $\alpha < \arg z < \beta$ will contain $P_n$, and that the $P_n$'s are disjoint. Thus the proof can be finished by showing that if $w \neq 0$ then $w \in f(P_n)$ for infinitely many $n$.

Let $i c_n$ be the midpoint of the interval in which $\frac{1}{2} \exp(P_n)$ meets the imaginary axis. Since $\frac{1}{2} \exp(P_n)$ is the interior of one component of the portion of the ribbon lying in the half plane $\text{Im } z > 0$, we know that $c_n > 0$.

**Lemma.** If $a > 0$ then $R_n(a, c) \subseteq \cosh(P_k)$ for infinitely many values of $k$.

Using this lemma we let $U = P_k$, where $k > n_0$ is chosen so large that $R_n(\log n_k, c_k) \subseteq \cosh(P_k)$ and $P_k$ is disjoint from $U$, ... $U_{n-1}$. Then $f(U) \subseteq \exp(R_n(\log n_k, c_k)) = A_n$, and, since every $w \neq 0$ lies in infinitely many $A_n$, $f^{-1}(w) \cap \{a < \arg z < \beta\}$ is infinite. Thus $\theta \in f[f]$. □

**Remark.** This proof actually shows that if $\theta \neq 0$ or $\pi/2$ (mod $\pi$) and if $N$ is any $\epsilon$-neighborhood of a ray from the origin through $e^{i\theta}$, then $f^{-1}(w) \cap N$ is infinite if $w \neq 0$. Had we merely wished to prove the Theorem we could have replaced the $P_n$'s with a sequence of disjoint rectangles $R_n = \{x + iy : a_n < x < b_n, 2mn < y < 2mn + \pi\}$ which lie in $\alpha < \arg z < \beta$ and for which the sequences $a_n$ and $b_n - a_n$ approach $\infty$. Then $\frac{1}{2} \exp(R_n)$ is the intersection of the half plane $\text{Im } z > 0$ with the annulus $\exp(a_n) < |z| < \exp(b_n)$. Since the width $\exp(a_n) - \exp(b_n)$ of the annulus approaches $\infty$, it is geometrically clear that if $a > 0$ and if $c_n = \frac{1}{2}(\exp(a_n) + \exp(b_n))$, there will exist infinitely many $n$'s for which the rectangle $R_n(a_n, c_n)$ lies inside $\frac{1}{2} \exp(R_n)$. But when $a_n$ is large the boundary of $\cosh(R_n)$ will stay very close to the boundary of $\frac{1}{2} \exp(R_n)$ because then $\frac{1}{2} \exp(-R_n)$ must lie within a tiny neighborhood of $0$. Thus $R_n(a_n, c_n) \subseteq \cosh(R_n)$ for infinitely many $n$'s. Using this version of the Lemma, the Theorem can be proved by replacing the parallelograms $P_n$ in the proof above with the rectangles $R_n$.

**Proof of the Lemma.** Suppose that $\delta > 0$ and let $b_n$ be the largest number such that the interior of $R_\delta(b_n, c_n)$ is contained in $\frac{1}{2} \exp(P_n)$. (Once the width of the ribbon exceeds $2\delta$, $b_n$ will be positive.) Then some vertex of $R_\delta(b_n, c_n)$ lies on $\gamma_+$ or $\gamma_-$, and we shall show that this implies the unboundedness of $\{b_n\}$. Since $|\gamma_\pm(y)|$ are increasing functions of $y$ there are just two cases: (1) the northeast vertex $b_n + i(c_n + \delta)$ lies on $\gamma_+$, or (2) the southwest vertex $-b_n + i(c_n - \delta)$ lies on $\gamma_-$. When case (1) holds we have...
(A) \( b_n + i(c_n + \delta) = \gamma_+(2\pi n + \phi_n) \) where \( \phi_n = \tan^{-1}[(c_n + \delta)/b_n] \).

Assuming that \( b_n \) is bounded implies that \( \phi_n \to \pi/2 \) because \( c_n \to \infty \).

Now we divide equation (A) by \( \gamma_+(2\pi n + \pi/2) \). Since

\[
\gamma_n = \frac{1}{2i} \left( \gamma_+ \left( 2\pi n + \frac{\pi}{2} \right) + \gamma_- \left( 2\pi n + \frac{\pi}{2} \right) \right)
\]

and since \( \frac{\gamma_-(y)}{\gamma_+(y)} = \exp \left[ \frac{-2\epsilon}{\lambda} \right] \)

the left side becomes

\[
\left[ \frac{b_n + i\delta}{\gamma_+(2\pi n + \pi/2)} \right] + \frac{1}{2} \left( 1 + \exp \left[ \frac{-2\epsilon}{\lambda} \right] \right).
\]

The right side becomes \( \exp \left[ (1/m + i)(\phi_n - \pi/2) \right] \) and if case (1) obtains for infinitely many \( n \) we can equate the limit as \( n \to \infty \) of each side and produce the contradiction \( \frac{1}{2} (1 + \exp [-2\epsilon/\lambda]) = 1 \).

Case (2) gives

(B) \( -b_n + i(c_n - \delta) = \gamma_- \left( 2\pi n + \pi - \psi_n \right) \) where \( \psi_n = \tan^{-1}[(c_n - \delta)/b_n] \).

If \( b_n \) is bounded and (B) holds infinitely often, then dividing by \( \gamma_- \left( 2\pi n + \pi/2 \right) \) and letting \( n \) approach \( \infty \) makes \( \psi_n \to \pi/2 \) and gives the contradiction \( \frac{1}{2} (\exp (2\epsilon/m) + 1) = 1 \).

This proves that \( b_n \) is unbounded. Thus, in particular, if \( \delta = \pi + 1 \) and \( b = a + 1 \) the inclusion \( R_n(a, c_n) \subset R_\delta(b, c_n) \subset \frac{1}{2} \exp (P_n) \) holds for infinitely many values of \( n \). Then when \( n \) is large enough the set \( \frac{1}{2} \exp (-P_n) \) will lie in a neighborhood of \( 0 \) so small that \( \cosh (P_n) \) very nearly contains \( R_\delta(b, c_n) \) and certainly contains \( R_n(a, c_n) \).

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REFERENCES

