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ON A SUBLATTICE OF THE LATTICE OF NORMAL FITTING CLASSES

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ABSTRACT. Let **L** be the set of all Fitting classes $\mathfrak{F}$ with the following two properties: (i) $\mathfrak{F} \supseteq \mathfrak{N}$, the class of all finite nilpotent groups, and (ii) every $\mathfrak{F}$-avoided, complemented chief factor of any finite soluble group G is partially $\mathfrak{F}$-complemented in G. It is shown that **L** is a complete sublattice of the complete lattice **N** of all nontrivial normal Fitting classes, and, moreover, it is lattice isomorphic to the subgroup lattice of the Frattini factor group of a certain abelian torsion group due to H. Lausch.

1. Introduction. All groups considered in this note are finite and soluble except when stated otherwise. A class $\mathfrak{X}$ of groups is called a *Fitting class* if (1) $N \in \mathfrak{X}$ whenever $G \in \mathfrak{X}$ and $N \trianglelefteq G$, and (2) $G \in \mathfrak{X}$ whenever G is the product of normal subgroups each of which belongs to $\mathfrak{X}$. Given a Fitting class $\mathfrak{F}$, a subgroup V of a group G is called an *$\mathfrak{F}$-injector* of G if, for any subnormal subgroup N of G, $V \cap N$ is an $\mathfrak{F}$-maximal subgroup of N. In view of Satz 1 of Fischer, Gaschütz and Hartley [3], every group G has a unique conjugacy class of $\mathfrak{F}$-injectors corresponding to every Fitting class $\mathfrak{F}$. For various interesting properties of injectors we refer the readers to [3] and Hartley [5].

As in [8], a Fitting class $\mathfrak{X}$ is said to have the property (Λ) if in each group G every $\mathfrak{X}$-avoided, complemented chief factor of G is necessarily a partially $\mathfrak{X}$-complemented chief factor. A chief factor of a group G is called *$\mathfrak{X}$-avoided* (*$\mathfrak{X}$-covered*) if it is avoided (covered) by an $\mathfrak{X}$-injector of G. A complemented chief factor of G is said to be *partially $\mathfrak{X}$-complemented* if at least one of its complements contains an $\mathfrak{X}$-injector of G. In [8], the author showed that a Fischer class (see Hartley [5], for the definition) has the property (Λ) if and only if it is the Fischer class of all solu-

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ble π-groups for some set π of primes. As pointed out there, one can have, in general, a Fitting class which has the property (Λ) but which is not the class of all soluble π-groups for any set π of primes. In fact, the (normal) Fitting class of all groups each of which acts by conjugation as an even permutation group on its largest normal $2'$-subgroup is an example of such a Fitting class. For the details of this Fitting class, we refer the readers to Blessenohl and Gaschütz [2].

As in [2], a Fitting class $\mathfrak{F}$ is called *normal* if in every group G an $\mathfrak{F}$-injector of G is a normal subgroup. In view of [2, Satz 6.2], the set $\mathbf{N}$ of all nontrivial normal Fitting classes is a complete lattice under the operations $\wedge$ and $\vee$ defined by

$$\mathfrak{F}_1 \wedge \mathfrak{F}_2 = \mathfrak{F}_1 \cap \mathfrak{F}_2 \quad \text{and} \quad \mathfrak{F}_1 \vee \mathfrak{F}_2 = \bigcap \{\mathfrak{X} \in \mathbf{N} \mid \mathfrak{F}_1, \mathfrak{F}_2 \subseteq \mathfrak{X}\} \quad \forall \mathfrak{F}_1, \mathfrak{F}_2 \in \mathbf{N}.$$

Recently, Lausch [7] has shown that this lattice $\mathbf{N}$ is lattice isomorphic to the subgroup lattice of a certain infinite abelian torsion group which he constructs in the course of proving Theorem 2.4 in [7]. More precisely, he shows (cf. [7, Corollary 2.5]) that if (A, d) is a normal Fitting pair (see [7] for the definition) admitted by the least element of $\mathbf{N}$, then $\mathbf{N}$ is lattice isomorphic to the subgroup lattice of A. Our aim in this note is to prove the following:

Theorem 1. *Let* $\mathbf{L}$ *be the set of all Fitting classes with the property* (Λ) *each of which contains the class* $\mathfrak{N}$ *of all finite nilpotent groups. Then*:

(1) $\mathbf{L}$ *is a complete sublattice of* $\mathbf{N}$; *and*

(2) *if* (A, d) *is a normal Fitting pair admitted by the least element of* $\mathbf{N}$, *then* $\mathbf{L}$ *is lattice isomorphic to the subgroup lattice of* $A/\Phi(A)$, *where* $\Phi(A)$ *denotes the Frattini subgroup of* A.

For the definition of $\Phi(A)$, we refer the readers to Fuchs [4, p. 35]. The above theorem is proved in §3, while in §2 we discuss a property of the elements of $\mathbf{L}$ on which hinges the proof of Theorem 1.

2. Fitting classes with the property (Λ). Our main result of this section is the following:

Theorem 2. *Every Fitting class with the property* (Λ) *which contains* $\mathfrak{N}$ *is normal.*

In order to prove this theorem we need the following result which is an extension of Lemma 1 of [1]. Once again we refer the readers to Huppert [6] and B. H. Neumann [10] for the definitions of the standard and the twisted wreath products, respectively.

Lemma 3. *Let A and B be two finite groups, $S \leq B$, α a homomorphism from S into the automorphism group of A, and T a right transversal of S in B. Let $A \sim_S B = [A^T]B$ be the twisted wreath product of A by B over S. If $T = XC$, where C is a subgroup of B and X is a subset of a subgroup R of B such that $R \cap C = \{1\}$, then $[A^T]C \cong A^X \sim_r C$.*

Proof. We observe first that if $f \in A^T$ then, for each $b \in B$ and each $t \in T$, $f^b(t) = \alpha(s^{-1})(f(t'))$, where t' and s are determined by the equation $tb^{-1} = st'$. Now, let ϕ be the mapping from $[A^T]C$ into $A^X \sim_r C$ which assigns to each element $(f, c) \in [A^T]C$ the element $(f^*, c) \in A^X \sim_r C$, where $f^*: C \to A^X$ is defined by $(f^*(d))(x) = f(xd)$ for each $d \in C$ and each $x \in X$. Then ϕ is well defined and, as in the proof of Lemma 1 of [1], ϕ is a monomorphism. It remains to show that ϕ is onto. To see this, let $(g, c) \in A^X \sim_r C$ and let $t \in T$. Since $T = XC$, $X \subseteq R \leq B$ and $R \cap C = \{1\}$, there exist unique elements $d \in C$ and $x \in X$ such that $t = xd$. Now, define a mapping $f \in A^T$ by $f(t) = (g(d))(x)$ for each $t \in T$. Then $\phi(f, c) = (g, c)$, and so, ϕ is onto, as required.

We can now prove Theorem 2.

Proof of Theorem 2. Let $\mathfrak{F}$ be a Fitting class with the property (Λ) and let $\mathfrak{F} \supseteq \mathfrak{N}$. In view of the main result in [9], it will be sufficient to show that $(G \times \cdots \times G) \sim_r C \in \mathfrak{F}$ (p factors in first term) whenever $G \in \mathfrak{F}$ and $C = \langle c \rangle$ is a cyclic group of order p, a prime. Let $R = \langle x \rangle$ be a cyclic group of order p^2, let $H = C \times R$ and let K be the subgroup of H generated by cx^p. Let W be the twisted wreath product of G by H over K with the action of K on G being the trivial one. If $X = \{1, x, \cdots, x^{p-1}\}$ and $T = XC$, the complex product, then T is a right transversal of K in H and, moreover, W is the semidirect product of G^T by H, with the action of H on G^T being as follows: For each $f \in G^T$, each $h \in H$ and each $t \in T$, $f^h(t) = (f(t'))$, where $t' \in T$ is determined by the equation $th^{-1} = kt'$ with $k \in K$. Thus, since H is abelian, K acts trivially on the base group G^T. In particular, $G^T \times K$ is contained in the $\mathfrak{F}$-injector V of W. But then we must have that G^TC/G^T is an $\mathfrak{F}$-covered chief factor of W, for, otherwise it would be an $\mathfrak{F}$-avoided, complemented chief factor of W which is not partially $\mathfrak{F}$-complemented in W since G^TK is not contained in any complement of G^TC/G^T in W. Thus, $V \geq G^TC$, and so, $G^TC \in \mathfrak{F}$ since $G^TC \lhd\lhd V$. However, by Lemma 3, $G^TC \cong G^X \sim_r C$. Hence, since G^X is isomorphic to a direct product of p

copies of G, it follows that $(G \times \cdots \times G) \sim_r C \in \mathfrak{F}$ (p factors in first term) and the theorem is proved.

We conclude this section with the remark that for a Fitting class $\mathfrak{F}$ with the property (Λ) which does not necessarily contain $\mathfrak{N}$, one can show, using the same techniques as those used in the proofs of the main result in [9] and Theorem 2 above, that the normalizer in any group G of an $\mathfrak{F}$-injector of G has index a π'-number, where π is the uniquely determined set of primes such that $\mathfrak{N} \cap \mathfrak{S}_\pi \subseteq \mathfrak{F} \subseteq \mathfrak{S}_\pi$, the class of all soluble π-groups (see Hartley [5, §3.3, Remark 1]).

3. Lattice properties of L. In this section, we will prove Theorem 1, our main result of this note. For the proof of the theorem we will need the following two lemmas of which the first one is a simple characterization of Fitting classes in L.

Lemma 4. *Let $\mathfrak{F}$ be a Fitting class which contains $\mathfrak{N}$. Then $\mathfrak{F} \in$ L if and only if $\mathfrak{F}$ is normal and the factor group of every group G with respect to its $\mathfrak{F}$-injector has elementary abelian Sylow subgroups.*

Proof. Suppose first that $\mathfrak{F} \in$ L. Then, by Theorem 2, $\mathfrak{F}$ is normal, and so, in view of Satz 5.3 of Blessenohl and Gaschütz [2], the factor group of every group with respect to its $\mathfrak{F}$-injector is abelian. Assume to the contrary that there exists a group K whose factor group K/V with respect to its $\mathfrak{F}$-injector V is not elementary abelian. Then, clearly K has a subgroup G such that $G > V$ and G/V is cyclic of order p^2, for some prime p. Let $x \in G$ be such that $\langle V, x\rangle = G$, let $H = \langle y\rangle$ be a cyclic group of order p and let $W = G \times H$. Then $V \times H$ is the $\mathfrak{F}$-injector of W, and, therefore, $V\langle x^p y\rangle \notin \mathfrak{F}$. In particular, $V\langle x^p y\rangle/V$ is an $\mathfrak{F}$-avoided chief factor of W which is, moreover, complemented in W. However, it is easy to check that $V\langle x^p y\rangle/V$ is not partially $\mathfrak{F}$-complemented in W, contrary to $\mathfrak{F}$ having the property (Λ). Hence, the assertion.

Suppose next that $\mathfrak{F}$ is normal and that the factor group of every group with respect to its $\mathfrak{F}$-injector is elementary abelian. Let G be an arbitrary group, V an $\mathfrak{F}$-injector of G and H/K an $\mathfrak{F}$-avoided chief factor of G. Since G/V is elementary abelian, VH/VK is clearly a complemented chief factor of G. But a complement of VH/VK is also a complement of H/K and, moreover, it contains V. Hence $\mathfrak{F} \in$ L and the proof is complete.

Next, we have

Lemma 5. *Let $\{\mathfrak{F}_i \mid i \in I\}$ be a family of Fitting classes. If $\mathfrak{F}_i \in$ L for* *each $i \in I$, then $\mathfrak{X} = \bigcap_{i \in I} \mathfrak{F}_i \in \mathbf{L}$. In particular, there exists a unique smallest Fitting class containing $\mathfrak{N}$ which has the property* (Λ).

Since each $\mathfrak{F}_i$ is, by Theorem 2, a normal Fitting class, the above lemma follows from Blessenohl and Gaschütz [2, Lemma 4, Satz 6.2] and the fact that the $\mathfrak{X}$-injector of a group G is the intersection of the $\mathfrak{F}_i$-injectors of G, $i \in I$.

Finally, we can prove our main result.

Proof of Theorem 1. The first part of the theorem, namely that $\mathbf{L}$ is a complete sublattice of $\mathbf{N}$, is immediate from Theorem 2, Lemma 5 and the fact that $\mathfrak{F} \in \mathbf{L}$ implies $\mathfrak{H} \in \mathbf{L}$ whenever $\mathfrak{H}$ is a Fitting class such that $\mathfrak{F} \subseteq \mathfrak{H}$. The latter fact follows easily from Lemma 4.

As for the second part of the theorem, let θ be the complete lattice isomorphism as defined in Lausch [7, Corollary 2.5] between $\mathbf{N}$ and the subgroup lattice of A, and let $(\theta|\mathbf{L})$ denote its restriction to the sublattice $\mathbf{L}$ of $\mathbf{N}$. Set $\phi = (\theta|\mathbf{L})\sigma$, where $\sigma\colon A \to A/\Phi(A)$ is the canonical epimorphism. Then ϕ is a complete lattice isomorphism from $\mathbf{L}$ onto the subgroup lattice of $A/\Phi(A)$. For, observe first that if $\mathfrak{F} \in \mathbf{L}$ then, as in the proof of Theorem 2.4 of Lausch [7], $(A/B, d\mu)$, where $B = \mathfrak{F}(\theta|\mathbf{L})$ and $\mu\colon A \to A/B$ is the canonical epimorphism, is a normal Fitting pair admitted by $\mathfrak{F}$. Thus, in view of Lemma 4 and the definition of a normal Fitting pair, we have $\Phi(A) \subseteq B$. The rest follows from the fact that θ is a complete lattice isomorphism.

It follows from the proof of the second part of Theorem 1 and Proposition 2.1 of Lausch [7] that

Corollary 6. *Let $\mathfrak{F} \in \mathbf{L}$ and let (B, d_1) and (A, d) be normal Fitting pairs admitted by $\mathfrak{F}$ and the least element of $\mathbf{N}$, respectively. Then, B is isomorphic to a factor group of $A/\Phi(A)$.*

We conclude this note with a remark that $\mathbf{L}$ is precisely the set of all Fitting classes each of which contains $\mathfrak{N}$ and has the property that in every group G the corresponding injectors cover all the Frattini chief factors of G. For, let $\mathfrak{F} \supseteq \mathfrak{N}$ be a Fitting class with this property, let $G \in \mathfrak{F}$ and let $C = \langle c \rangle$ be a cyclic group of order p^2. If $W = G \sim_r C = [G^C]C$, then clearly $[G^C]\langle c^p \rangle \in \mathfrak{F}$. But

$$[G^C]\langle c^p\rangle \cong \underbrace{(G \times \cdots \times G)}_{\leftarrow p \text{ factors} \rightarrow} \sim_r C_p,$$

where C_p is a cyclic group of order p. Thus, $\mathfrak{F}$ has the wreath product property, and so, by the main result of [9], $\mathfrak{F}$ is a normal Fitting class. It is now easy to see that the factor group of every group G with respect to its $\mathfrak{F}$-

injector is elementary abelian. Hence, $\mathfrak{F} \in$ L. Conversely, if $\mathfrak{F} \in$ L, then in view of Theorem 2 and Corollary 6, the $\mathfrak{F}$-injector of any group G covers all the Frattini chief factors of G.

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