AN INEQUALITY FOR ANALYTIC FUNCTIONS

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ABSTRACT. If $F$ denotes the boundary value of a function $f \in H^p$, $1 \leq p \leq \infty$, the infimum of the measure of $\left\{ \theta \mid |F(\theta)| > A \right\}$ for given $A$, $0 < A < |f(0)|$, $\|f\|_{H^p} = 1$, is determined.

In this note we discuss an aspect of the boundary behaviour of certain functions analytic on the unit disc in terms of their values at the origin. Specifically, if $F(\theta)$ is the boundary value of a function $f \in H^p$, $1 < p < \infty$, and if a number $A$, $0 < A < |f(0)|$, is chosen, we are interested in minimizing the measure of $\{ \theta \mid 0 \leq \theta \leq 2\pi$ and $|F(\theta)| > A \}$ over all $f \in H^p$ with $\|f\|_{H^p} = 1$.

The result is that, for $p < \infty$, the infimum is the solution $c$ of the equation

$$(1) \quad c \log \left[ 1 + 2\pi(1 - A^p)/cA^p \right] = 2\pi p \log (|f(0)|/A),$$

while if $p = \infty$, the infimum is $2\pi(1 - \log |f(0)|/\log A)$, which is the limit, as $p \to \infty$, of the solutions of (1).

The ingredients of the proof are Jensen's inequality and Jensen's formula which we state in the following forms.

1. Jensen's inequality. If $\mu$ is a positive measure on a measure space $X$ with $\mu(X) = 1$, and if $f \in L^1(d\mu)$, then

$$\int_X |f| d\mu \leq \log \int_X |f| d\mu.$$  

Equality holds if, and only if, $f$ is an outer function [1, p. 62].

2. Jensen's formula. If $f \in H^1$ of the unit disc and $F(\theta) = \lim_{r \to 1} f(re^{i\theta})$, then

$$\log |f(0)| \leq \frac{1}{2\pi} \int_0^{2\pi} \log |F(\theta)| d\theta.$$  

Equality holds if, and only if, $f$ is an outer function [1, p. 62].

We begin by showing, in Lemma 1, that for each pair of numbers $A_0$, $A$, $0 < A < A_0 < 1$, there is a step function $b$ on $(0, 2\pi)$ satisfying $\int_0^{2\pi} |b| = 2\pi$ and $\int_0^{2\pi} \log |b| = 2\pi \log A_0$, and which takes on the value $A$ and one other.

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value \( \bar{y} \). For \( H^1 \), the number \( c = \text{measure}\{x| h(x) = \bar{y}\} \) will be the required solution of (1).

\textbf{Lemma 1.} Given \( 0 < A < A_0 < 1 \), there exist unique numbers \( c \) and \( \bar{y} \) (depending on \( A \) and \( A_0 \)) such that \( 0 < c < 2\pi \), \( 1 < \bar{y} \) and

\( 2\pi = (2\pi - c)A + c\bar{y} \) and \( 2\pi \log A_0 = (2\pi - c) \log A + c \log \bar{y} \).

Further, \( c \) satisfies the equation

\( c \log \frac{1 + 2\pi(1 - A)/cA}{1 - A} = 2\pi \log (A/A_0) \).

Also, for fixed \( A \), the left-hand side of (3) is an increasing function of \( c \).

\textbf{Proof.} If \( \bar{y} \) is the unique solution of

\( y - A = \frac{\log y - \log A}{1 - A} \log A_0 - \log A \)

for \( y > A \), then \( \bar{y} > 1 \). Let

\( c = \frac{2\pi(1 - A)}{(\bar{y} - A)} \).

Since equations (4') with \( y = \bar{y} \) and (4''), together, are equivalent to equations (2), the pair \( c, \bar{y} \) satisfies (2). Then from (4''), \( 0 < c < 2\pi \).

Equation (3) follows easily from (2) and a routine calculus argument proves the last statement of the lemma.

\textbf{Lemma 2.} If \( 0 < c < 2\pi \), \( G \in L^1(0, 2\pi) \) and \( |G| \leq A < 1 \) on a set \( S \) of measure \( 2\pi - c \), then

\( (2\pi - c)A - \int_S |G| \leq (2\pi - c) \log A - \int_S \log |G| \).

\textbf{Proof.} This follows immediately from the inequality \( \log A - A \geq \log t - t \) for \( 0 < t \leq A < 1 \).

\textbf{Lemma 3.} Suppose \( 0 < A < A_0 < 1 \) and \( c, \bar{y} \) are the constants satisfying equations (2) of Lemma 1. Let \( F \in L^1(0, 2\pi) \) with \( (1/2\pi) \int_0^{2\pi} |F| = 1 \) and \( (1/2\pi) \int_0^{2\pi} \log |F| = \log A_0 \). If \( |F| \leq A \) on a set \( S \) of measure \( 2\pi - c \), then \( |F| = A \) on \( S \) and \( |F| = \bar{y} \) off \( S \).

\textbf{Proof.} Let \( S' = [0, 2\pi] - S \). Then

\[ \int_S |F| + \int_{S'} |F| = 2\pi \quad \text{and} \quad \int_S \log |F| + \int_{S'} \log |F| = 2\pi \log A_0. \]

From equations (2), \( c \) and \( \bar{y} \) satisfy

\( 2\pi = (2\pi - c)A + c\bar{y} \) and \( 2\pi \log A_0 = (2\pi - c) \log A + c \log \bar{y} \).

Hence
\[ \int_{S'} |F| = (2\pi - c)A + c\bar{y} - \int_{S} |F| \]

and
\[ \int_{S'}, \log |F| = (2\pi - c) \log A + c \log \bar{y} - \int_{S} \log |F|. \]

Let
\[ a = (2\pi - c)A - \int_{S} |F| \quad \text{and} \quad b = (2\pi - c) \log A - \int_{S} \log |F|. \]

Then
\[ \int_{S'}, |F| = a + c\bar{y} \quad \text{and} \quad \int_{S'}, \log |F| = b + c \log \bar{y}; \]

and also from Lemma 2, \(0 < a \leq b\). Now, by Jensen's inequality
\[ \int_{S'}, \log |F| \frac{d\theta}{c} \leq \log \int_{S'} |F| \frac{d\theta}{c}, \]

so that \(a/c + \log \bar{y} \leq b/c + \log \bar{y} \leq \log(a/c + \bar{y})\). Since \(\bar{y} > 1\), it follows that \(a = b = 0\).

Therefore,
\[ \int_{S'}, |F| \frac{d\theta}{c} = \bar{y} \quad \text{and} \quad \int_{S'}, \log |F| \frac{d\theta}{c} = \log \bar{y}, \]

and the uniqueness part of Jensen's inequality implies \(|F| = \bar{y}\) on \(S'\).

Also, since \(a = b = 0\), equations (5) imply
\[ \int_{S} |F| = (2\pi - c)A \quad \text{and} \quad \int_{S} \log |F| = (2\pi - c) \log A, \]

and so for the same reason, \(|F| = A\) on \(S\).

**Notation.** If \(S\) is a subset of \([0, 2\pi]\), \(m(S)\) will denote the Lebesgue measure of \(S\).

**Theorem A.** Let \(f \in H^1\) and \(|f|_{H^1} = 1\). Suppose \(0 < A < |f(0)|\) and let \(c\)
be the solution of
\[ c \log \left[1 + 2\pi(1 - A)/cA\right] = 2\pi \log \left(|f(0)|/A\right). \]

If \(F(\theta) = \lim_{r \uparrow 1} f(re^{i\theta})\), then \(m(\theta| |F(\theta)| > A|)\) is greater than \(c\).

**Proof.** If \(f\) is an outer function, then \((1/2\pi) \int_{0}^{2\pi} |F| = 1\) and by Jensen's formula,
\[ \frac{1}{2\pi} \int_{0}^{2\pi} \log |F| = \log |f(0)|. \]

Therefore, by Lemma 3, \(m(\theta| |F(\theta)| > A|)\) is bigger than \(c\).

If \(f\) is not an outer function, write \(f = BSg\), where \(B\) is the Blaschke product of the zeros of \(f\), \(S\) is the singular part of \(f\), and \(g\) is an outer function \([1]\). If \(G(\theta) = \lim_{r \uparrow 1} |g(re^{i\theta})|\), then \(|G(\theta)| = |F(\theta)|\) a.e., so that
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\( m(\{|G| > A\}) = m(\{|F(0)| > A\}) \). Also since \( g \) is an outer function we have \( |g(0)| > |f(0)| \). By Lemma 1, we have that the solution \( c \) of \( c \log \left[ 1 + 2\pi(1 - A)/cA \right] = 2\pi \log (|g(0)|/A) \) is bigger than the solution \( c \) of (6) concluding the proof.

We remark that the number \( c \) satisfying (6) is the best possible, since we can find \( g \in H^1 \) such that \(|G| \) is arbitrarily close to the step function \( H: H(\theta) = A, 0 \leq \theta < 2\pi - c, H(\theta) = \overline{A}, 2\pi - c \leq \theta < 2\pi \).

Now, if \( f \in H^p, 1 \leq p < \infty \), then \( f^p \in H^1 \) and \( m(\{|F(0)| > A\}) = m(\{|F| > A^p\}) \). This reduces to the \( H^1 \) case with \( f \) replaced by \( f^p \) and \( A \) by \( A^p \). Hence,

**Theorem B.** Let \( f \in H^p, 1 \leq p < \infty \), and \( \|f\|_{H^p} = 1 \). If \( 0 < A < |f(0)| \), then \( m(\{|F(0)| > A\}) \) is greater than the solution \( c \) of

\[
 c \log \left[ 1 + 2\pi(1 - A^p)/cA^p \right] = 2\pi p \log (|f(0)|/A)
\]

and this is the best possible.

Finally, for \( H^\infty \) (or the disc algebra), if \( f \in H^\infty \) and \( \|f\|_{\infty} \leq 1 \), we have

\[
 2\pi \log |f(0)| \leq \int_{|F| \geq A} \log |F| + \int_{|F| > A} \log |F| \leq \int_{|F| \leq A} \log |F|
\]

since \( \log |F| < 0 \).

Therefore, \( 2\pi \log |f(0)| \leq \log A \cdot m(\{|F(0)| \leq A\}) \), or

\[
 2\pi \frac{\log |f(0)|}{\log A} \geq m(\{|F(0)| \leq A\}) = 2\pi - m(\{|F(0)| > A\}),
\]

or

\[
 m(\{|F(0)| > A\}) \geq 2\pi(1 - \log |f(0)|/\log A).
\]

Again it is easy to see that this is the best possible.

**Remarks.** (I) The same methods apply, as well, to any positive measure \( \mu \) on a measure space \( X \) for which Jensen's formula \( \log \int_X |f|d\mu \leq \int_X \log |f|d\mu \) holds. (See \cite{1}, pp. 54–57, for some examples.) For these cases, equations (1) and (7) are valid if \( 2\pi \) is replaced by \( \mu(X) \).

(II) By considering the Poisson kernel, it follows from (I) that we can solve the same problem in the unit disc knowing \( f \) at some point \( z^* \) other than 0. A particular result is that, if \( f \in H^\infty \) of the unit disc, \( F \) is the boundary value of \( f \), \(|z^*| < 1 \) and \( 0 < A < |f(z^*)| \), then the Lebesgue measure of \( \{|\theta| \ F(\theta) \leq A\} \) is less than or equal to
\[ 4 \tan^{-1} \left[ \frac{1 + |z'|}{1 - |z'|} \tan \frac{\pi}{2} \frac{\log |f(z')|}{\log A} \right]. \]

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REFERENCE


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