ON THE FAILURE OF THE FIRST PRINCIPLE OF
SEPARATION FOR COANALYTIC SETS

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ABSTRACT. In this note we present a new example of a pair of disjoint
cancoanalytic sets which are not Borel separable, i.e., coanalytic sets \( D \) and
\( H \) such that \( D \cap H = \emptyset \) and such that there is no Borel set \( E \) for which \( D \subseteq E \) and \( E \cap H = \emptyset \).

1. There are in the literature several proofs of the existence of a pair
of disjoint coanalytic sets which are not Borel separable [3, pp. 220, 260,
263], [4, p. 25], [5]. In this note we present yet another proof. We show
that Blackwell's construction in [1] of a Borel set which does not admit a
Borel uniformization yields explicitly a pair of disjoint coanalytic sets
which are not Borel separable.

2. First, we briefly recall Blackwell's construction. Let \( U \) be the
set of all finite sequences of positive integers of positive length. Let \( X \)
be the power-set of \( U \). Identify \( X \) with \( 2^U \) and endow \( X \) with the prod-
uct of discrete topologies, so that \( X \) is a homeomorph of the Cantor set.
With each \( x \in X \), associate a game \( G(x) \) between players \( \alpha \) and \( \beta \) as fol-
lows: the players alternately choose positive integers, \( \alpha \) choosing first,
each choice being made with complete information about all previous choices.
For any play \( \omega = (n_1, n_2, \ldots, \) \), let \( k(\omega) \) be the first \( i \) such that
\( (n_1, n_2, \ldots, n_i) \notin x \), and let \( k(\omega) = \infty \) if \( (n_1, n_2, \ldots, n_i) \in x \) for all \( i \). A
play \( \omega \) is a win for \( \alpha \) in \( G(x) \) just in case \( k(\omega) \) is even, it is a win for \( \beta \)
if \( k(\omega) \) is odd, and it is a draw if \( k(\omega) = \infty \). In any game \( G(x) \), the space
\( Y_1 \) of strategies for \( \alpha \) can be identified with the set \( N^N \) of (infinite) se-
quencies of positive integers, which we equip with the product of discrete
topologies. A similar remark applies to the space $Y_2$ of strategies for $\beta$.

Let $Y$ be the disjoint union of $Y_1$ and $Y_2$ and give $Y$ the union topology, so that $Y$ is a homeomorph of $NN$. Finally, let $B_1$ be the set of $(x, y) \in X \times Y_1$ such that $y$ ensures $\alpha$ at least a draw in $G(x)$, and let $B_2$ be the set of $(x, y) \in X \times Y_2$ such that $y$ ensures $\beta$ at least a draw in $G(x)$.

Then, Blackwell has proved that

(i) $B_1, B_2$ are Borel subsets of $X \times Y$ (indeed, $B_1, B_2$ are closed subsets of $X \times Y$);

(ii) $\pi(B_1 \cup B_2) = X$, where $\pi$ denotes projection of $X \times Y$ to $X$;

(iii) with each $x \in X$, it is possible to associate $x' \in X$ and an ordered pair $(A_1, A_2)$ of nonempty analytic subsets of $X$ in such a way that

(a) $G(x')$ is a win for $\alpha$ if $x \in A_1 - A_2$,

(b) $G(x')$ is a win for $\beta$ if $x \in A_2 - A_1$,

(c) the mapping $x \mapsto x'$ is Borel measurable, and

(d) for every ordered pair $(A_1, A_2)$ of nonempty analytic subsets of $X$, there is $x \in X$ such that $(A_1, A_2)$ is associated with $x$.

We shall say that "$x$ codes $(A_1, A_2)$" in case $(A_1, A_2)$ is associated with $x$.

3. We are now ready to state our example. Let $D = X - \pi(B_1)$, and let $H = X - \pi(B_2)$. In other words, $D$ is the set of $x \in X$ such that $G(x)$ is a win for $\beta$, while $H$ is the set of $x$'s such that $G(x)$ is a win for $\alpha$. We claim that $D, H$ are disjoint coanalytic sets which are not Borel separable.

From (i) it follows that $D$ and $H$ are coanalytic. (ii) implies that $D \cap H = \emptyset$. Next we note that $D, H$ are nonempty. To see this, let $x$ be the singleton set whose only member is the ordered pair $(1, 2)$. Plainly, $G(x)$ is a win for $\beta$, so that $x \in D$. A similar argument shows that $H \neq \emptyset$.

Now assume by way of contradiction that there is a Borel set $E \subseteq X$ such that $D \subseteq E$ and $E \cap H = \emptyset$. Plainly, $E$ and $X - E$ are both nonempty. Let $W = \{x \in X : x' \in E\}$. By (iii)(c), $W$ is Borel. We now assert that both $W$ and $X - W$ are nonempty. To see, for instance, that $W \neq \emptyset$, choose $x_0 \in X$ and a nonempty analytic set $C \subseteq X$ such that $x_0$ codes $(C, X)$ and $x_0 \notin C$ (we can do this; for, if not, then for each $z \in X$, $z$ codes $(\{z\}, X)$, so that there are no codes left for the other pairs). Since $x_0 \in X - C$, it follows from (iii)(b) that $G(x_0')$ is a win for $\beta$, so $x_0' \in D$ and hence, $x_0 \in W$. One shows $X - W \neq \emptyset$ similarly.

Consequently, from (iii)(d), there is $x^* \in X$ such that $x^*$ codes $(W, X - W)$. We now have:

$x^* \in E \rightarrow x^* \in W \rightarrow G(x^*)$ is a win for $\alpha \rightarrow x^* \in H \rightarrow x^* \notin E$; and also,
$x^{*'} \notin E \rightarrow x^* \in X - W \rightarrow G(x^{*'})$ is a win for $\beta \rightarrow x^{*'} \in D \rightarrow x^{*'} \in E$, so that $x^{*'} \in E \iff x^{*'} \notin E$, which yields the desired contradiction.

4. Finally, by using an argument due to Novikov [4, p. 25], one can deduce from the fact that $D$ and $H$ are not Borel separable that the set $B = B_1 \cup B_2$ does not admit a Borel uniformization. Indeed, suppose that $S$ is a Borel uniformization of $B$. Let $T = \pi(S - B_1)$. Since $\pi$ is continuous and one-one on $S$, $T$ is Borel [2, p. 487]. Now verify that $D \subseteq T$ and $T \cap H = \emptyset$, which contradicts the fact that $D$ and $H$ are not Borel separable.

BIBLIOGRAPHY


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