NORMAL MOORE SPACES IN
THE CONSTRUCTIBLE UNIVERSE

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ABSTRACT. Assuming the axiom of constructibility, points in
closed discrete subspaces of certain normal spaces can be simulta-
neously separated. This is a partial result towards the normal Moore
space conjecture.

The normal Moore space conjecture states that every normal Moore space
is metrizable. This is known to be not provable from the usual axioms of
set theory, since Silver [4] shows that Martin’s axiom with the negation of
CH implies the existence of a separable nonmetrizable normal Moore space.
In this paper we consider the situation under Gödel’s axiom of constructi-
bility (V = L).

Bing [1] shows that a normal Moore space is metrizable iff it is collec-
tionwise normal. Moore spaces have character \( \aleph_0 \) (i.e. are first countable).
The following is then a partial result towards proving the normal Moore space
conjecture in L.

**Theorem (V = L).** If \( X \) is a normal Hausdorff space of character \( \leq \aleph_1 \),
then \( X \) is collectionwise Hausdorff.

**Definition.** A space is collectionwise Hausdorff (CWH) iff every closed
discrete set of points can be simultaneously separated by disjoint open sets.
Let \( \text{CWH}(\kappa) \) be CWH restricted to sets of cardinality \( \leq \kappa \).

**Remarks.** 1. The Theorem shows consistent Bing’s conjecture [2] that
normal Moore spaces be CWH. It shows in fact that in a normal character \( \aleph_1 \)
space, a discrete collection of countable closed sets can be simultaneously
separated by the device of shrinking each closed set to a point, which then
is of character \( \aleph_1 \).

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1 While working on this paper, the author was a National Science Foundation
Graduate Fellow.
2. Bing [1] shows that there always is a normal space of character $2^{\aleph_1}$ which is not $\text{CWH}$.

3. Tall [4] shows the consistency of $\text{CWH}(\kappa)$ for all $\kappa < \aleph_\omega$, using forcing rather than $L$.

4. The author has constructed a normal $\text{CWH}$, not collectionwise normal space of character $c$.

5. The Theorem implies that every locally compact normal Moore space is metrizable.

6. Kunen asks whether $V = L$ can be weakened to $\text{GCH}$.

Let us fix $X$, a normal Hausdorff space of character $\aleph_1$. We will prove $\text{CWH}(\kappa)$ for all $\kappa$ by induction on $\kappa$. $\text{CWH}(\aleph_0)$ is a consequence of the regularity of $X$. The same argument, using normality, yields:

A countable, closed, discrete collection of closed sets can be simultaneously separated by disjoint open sets.

For singular $\kappa$ of cofinality $\aleph_0$ ($\text{cf}(\kappa) = \aleph_0$), $\text{CWH}(\kappa)$ follows from (*) and the induction hypothesis. Henceforth, we will implicitly assume $\text{CWH}(\lambda)$ for $\lambda < \kappa$.

**Definition.** A set $C \subseteq \kappa$ is *cub* (closed, unbounded) iff $C$ is unbounded and closed in the order topology. A set $A \subseteq \kappa$ is *stationary* (alternatively, Mahlo) iff $A \cap C \neq \emptyset$ for all cub $C$. It is easy to show that the intersection of 2 (in fact, of $< \text{cf}(\kappa)$) cub sets is cub, hence the intersection of a cub set and a stationary set is stationary.

Cub sets can also be characterized as follows. For $B \subseteq \kappa$, define $B^*(\alpha)$ to be the greatest $\beta \in B$ such that $\beta \leq \alpha$, if such a $\beta$ exists. Define $\alpha \sim_B \beta$ iff $\alpha \cap B = \beta \cap B$ so that $B^*(\alpha) = B^*(\beta)$ if defined. Then $B$ is closed iff $B^*(\alpha)$ is always defined, and $B$ is unbounded iff each $\sim_B$ equivalence class has cardinality $< \kappa$.

To simplify notation, assume that $\kappa$ is an arbitrary closed discrete subset of $X$, so that if we separate $\kappa$ we may conclude $\text{CWH}(\kappa)$. For $f : \kappa \to \omega_1$, let $f^*(\alpha)$ be the $(\alpha)$th basis set at the point $\alpha$. Then any separation $U, V$ of $H, K \subseteq \kappa$ has a refinement that is coded by an $f : \kappa \to \omega_1$. Let $W^f_\alpha = \bigcup f^*(\beta) : \beta < \alpha$.

We will go through $\kappa$ assigning points to $H$ and $K$, considering initial segments $W^f_\alpha$ of potential separations, and destroying them if we can. For regular $\kappa$ we will assume that $\kappa \subseteq X$ witnesses $\forall \text{CWH}(\kappa)$, and conclude from Lemma 1 that each $f$ can be destroyed on a stationary set. Then using the combinatorial principle (Lemma 2) to tell us which $f$ to consider at stage $\alpha$,
we will, in Lemma 3, define $H$ and $K$ that cannot be separated—contradicting the normality of $\chi$, thus establishing $CWH(\kappa)$.

**Lemma 1.** Either $A_f = \{ \alpha < \kappa : \overline{W}^f_\alpha \cap \kappa \neq \alpha \}$ is stationary for all $f : \kappa \to \omega_1$ or the points in $\kappa$ can be simultaneously separated.

**Proof.** If $A_f$ is not stationary, there is a cub $C$ such that $\alpha \in C$ implies $\overline{W}^f_\alpha \cap \kappa = \alpha$. This means that there is a $g$ so that $g^\#: (\alpha) \cap f^\#: (\beta) = \emptyset$ for all $\beta < C^\#: (\alpha)$. By the induction hypothesis there is an $h$ so that $h^\#: (\alpha) \cap g^\#: (\beta) = \emptyset$. Then $f^\#: (\alpha) \cap g^\#: (\alpha) \cap h^\#: (\alpha)$ separates the points of $\kappa$.

Define a stationary system to be a function $A : \kappa \lambda \to P(\kappa)$ such that

$$
(**) \quad f \uparrow \alpha = g \uparrow \alpha \to A_f \cap (\alpha + 1) = A_g \cap (\alpha + 1).
$$

By $(**)$, $\alpha \in A_f$ is meaningful whenever $\text{dom}(f) \supseteq \alpha$. If $\kappa$ cannot be separated, then the $A$ defined in Lemma 1 is a stationary system.

**Lemma 2 (V = L).** If $A$ is a stationary system and $\kappa$ is regular, then there is a $\Phi : S \subset \kappa \to \bigcup \{ a^\alpha, \alpha < \kappa \}$ such that:

(i) $\Phi(\alpha) : \alpha \to \alpha$.

(ii) For all $f \in \kappa \lambda$, $B_f = \{ \alpha : \Phi(\alpha) = f \uparrow \alpha \}$ is a stationary subset of $A_f$.

If $A_f = A$ for all $f \in \kappa \lambda$, then Lemma 2 is Jensen's $Q(A)$ [3]. We follow Jensen's proof, which uses the ideas of elementary submodel, transitive collapse, and the condensation lemma for $L$. The idea of the proof is as follows. $L$ has a very nice well ordering $\prec$. We define $\Phi(\alpha) = f \in a^\alpha$ where $(f, C)$ is the $\prec$-first such that $\alpha \in A_f$, $C$ is cub $\subset \alpha$, and $B_f \cap C = \emptyset$. If there were a counterexample to the lemma, there would be a $\prec$-least one $(f_0, C_0)$. With ordinary set theory it is easy to show that for a cub of $a$'s, $f_0 \uparrow \alpha \in a^\alpha$ and $C_0 \cap \alpha$ is cub in $\alpha$. Moreover, assuming $V = L$, we get $(f_0 \uparrow \alpha, C_0 \uparrow \alpha)$ is the $\prec$-first counterexample for a cub $C'$ of $a$'s. But then $B_{f_0} = C' \cap A_{f_0}$ is stationary, a contradiction.

**Proof.** Define $\Phi(\alpha)$ by induction on $\alpha$. $\Phi(\alpha) = f$ where $(f, C)$ is the $\prec$-first pair (if there is one; if there is none, leave $\Phi(\alpha)$ undefined) such that

1. $f \in a^\alpha$, $C$ is cub in $\alpha$.

2. If $(g, D) < (f, C)$ satisfies 1, then there is $\beta \subset \alpha$ such that $\Phi(\beta) = g \uparrow \beta$ and $\beta \in D \cap A_g$.

3. $\alpha \in A_f$.

We claim $\Phi$ works. If not, let $(f_0, C_0)$ be the $\prec$-first counterexample.
Define a continuous monotone increasing sequence $M_\alpha, \alpha < \kappa$, of elementary submodels of $L_{\kappa^++}$ containing $A$ and $\langle f_0, C_0 \rangle$, such that $M_\alpha \cap \kappa \in \text{OR}$ and $\text{card } M_\alpha < \kappa$. Let $\pi_\alpha$ be the transitive collapse of $M_\alpha$, and let $\delta(\alpha)$ be $\pi_\alpha(\kappa)$. Then

(a) $C' = \{\delta(\alpha); \alpha < \kappa\}$ is cub in $\kappa$,
(b) for $f \in K \cap M_\alpha, \pi_\alpha(f) = f \upharpoonright \delta(\alpha),$
(c) for $C \in P(\kappa) \cap M_\alpha, \pi_\alpha(C) = \delta(\alpha) \cap C$.

Thus for $\alpha \in \mathcal{P}', \langle f_0 \cap \alpha, C_0 \cap \alpha \rangle$ is $\langle\text{-}\text{least such that } 1 \text{ and } 2 \text{ hold. If} \alpha \in C' \cap A_{f_0}, 3 \text{ holds as well, so } \Phi(\alpha) = f_0 \upharpoonright \alpha$, a contradiction.

**Lemma 3** ($\forall = L$). CWH($\lambda$) for all $\lambda < \kappa$ implies CWH($\kappa$) for regular $\kappa$.

**Proof.** Define $H, K, U^f, V^f$ by induction on $\alpha$.

$$U^f_\alpha = \bigcup \{ f \upharpoonright \beta; \beta < \alpha, \beta \in H \}, \quad V^f_\alpha = \bigcup \{ f \upharpoonright \beta; \beta < \alpha, \beta \in K \}.$$ 

Then $W^f_\alpha = U^f_\alpha \cup V^f_\alpha$. Suppose $\beta$ has been placed in $H$ or $K$ for all $\beta < \alpha$.

Put $\alpha$ in $H$ unless $\Phi(\alpha)$ is defined, in which case $W^f_\alpha \cap \kappa - \alpha \neq \emptyset$ and has a least element $\nu(\alpha)$. If $\nu(\alpha) \in U^f_\alpha$, place $\nu(\alpha)$ in $K$; place it in $H$ otherwise. Put $\beta \in \nu(\alpha) - \alpha$ in $H$. By the regularity of $\kappa, \{ \alpha; \beta < \alpha \rightarrow \nu(\beta) \text{ is not defined} \text{ or } \nu(\beta) < \alpha \}$ is a cub $C''$.

If $H$ and $K$ could be separated, they could be separated by some $f''$. By Lemma 2, there is an $\alpha \in C''$ such that $\Phi(\alpha) = f \upharpoonright \alpha$. Then, if $\nu(\alpha) \in H$, it is a limit of points in the open sets of the cover $f''$ of $K$, and similarly with $H$ and $K$ interchanged. $H$ and $K$ are closed sets that cannot be separated. This contradiction follows from the assumption $\neg CWH(\kappa)$.

One of the ways that the regularity of $\kappa$ is vitally used in the above proof is in the proof of Lemma 3, where $C''$ shows that our arbitrary ordering of a subset of $X$ does not matter. We will use the following definitions to consider various orders on subsets of singular $\kappa$. Let $S \subset \kappa \subset X$, and let $\rho: S \rightarrow \kappa$ be one-to-one. The following depend on $S$ and $\rho$. Redefine $W^f_\alpha$ as $\bigcup \{ f \upharpoonright \beta; \rho(\beta) < \alpha \text{ and } \beta \in S \}$, and let $D^f_\alpha = W^f_\alpha \cap \{ \beta; \rho(\beta) \geq \alpha \text{ and } \beta \in S \}$. Call $f$ thick wrt $S, \rho$ if there is an $\alpha$ such that $\text{card } D^f_\alpha > \alpha$; $f$ is thin if not.

**Lemma 4** (GCH). For every $S, \rho$ there is an $f$ thin wrt $S, \rho$.

**Proof.** If not, we can repeat the argument of Lemma 3 to show $X$ is not normal. Define $H$ and $K$ by induction using $\rho$ and the Gödel pairing function $\alpha \mapsto \langle \alpha_1, \alpha_2 \rangle$, which has the property that $\alpha < \max(\alpha_1^+, \alpha_2^+)$. At stage $\alpha$ consider (if meaningful) the $\alpha_1$th function $f: \alpha_2 \rightarrow \omega_1$, and assign (if possible)
a new point from $D^{f}_{a_{2}}$ to $H$ or $K$ to destroy $f$. Assign $\rho^{-1}\alpha$ arbitrarily if it exists and is not already assigned. No $g^{\#}$ separates $H$ and $K$, for at stage $\alpha$ only card $\alpha$ points have been assigned, so by thickness and GCH (Gödel: $V = L \rightarrow \text{GCH}$), there was some $\alpha$ for which it was meaningful and possible to destroy $g$.

We need to choose $S, \rho$ carefully. Because $\kappa$ is singular, there is a set $C = \{c_{\alpha}: \alpha < \text{cf}(\kappa)\}$ of cardinals $> \text{cf}(\kappa)$ cub in $\kappa$ of order type $\text{cf}(\kappa)$. Let $B_{\alpha} = \{\gamma: \gamma = \text{cf}(\kappa) \cdot \delta + \alpha \text{ for some } \delta < c_{\alpha}\}$.

Then $B_{\alpha}$'s are disjoint subsets of $\kappa$ such that $B_{\alpha} \subset c_{\alpha}$, and card $B_{\alpha} = c_{\alpha}$.

**Lemma 5 (GCH).** $\text{CWH}(\lambda)$ for all $\lambda < \kappa$ implies $\text{CWH}(\kappa)$ for $\kappa > \text{cf}(\kappa) > \aleph_{0}$.

**Proof.** Define $S_{i}, \rho_{i}, f_{i}$ for $i \in \omega$, inductively. Let $S_{0} = \kappa, \rho_{0}$ the identity. Let $f_{i}$ be thin wrt $S_{i}, \rho_{i}$. Define $S_{i+1}^{f} = \bigcup\{D^{f}_{a}: \alpha \in C\}$. Then Lemma 1 separates $S_{i} = S_{i+1}^{f}$. Using this and $(*)$ we can separate $\bigcup\{S_{i} - S_{i+1}^{f}: i \in \omega\}$. It is sufficient to show $\bigcup\{S_{i} - S_{i+1}^{f}: i \in \omega\} = \kappa$, or equivalently, $\bigcap\{S_{i}: i \in \omega\} = \emptyset$. Let $\rho_{i+1}'$ map $D^{f}_{i}$ one-to-one to $B_{a}$, and let $\rho_{i+1}$ be a single valued $\rho_{i+1}'$. Then for $\beta \in D^{f}_{a}, \rho_{i}(\beta) > \alpha > \rho_{i+1}(\beta)$. Since there are no infinite descending sequences of ordinals, $\bigcap S_{i} = \emptyset$.

This completes the proofs of the induction steps, so the Theorem is proved.

REFERENCES


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