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SUBRINGS OF NOETHERIAN RINGS

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ABSTRACT. Let S be a subring of a ring R such that R is a finitely generated right S-module. Clearly, if S is a right Noetherian ring then so is R. Generalizing a result of P. M. Eakin, we show that if R is right Noetherian and S is commutative then S is Noetherian. We also show that if R_S has a finite generating set $\{u_1, \cdots, u_m\}$ such that $u_iS = Su_i$ for $1 \le i \le m$, then a right R-module is Noetherian, Artinian or semisimple iff it is respectively so as a right S-module. This yields a result of Clifford on group algebras.

Let S be a subring of a ring R such that R is finitely generated as a right S-module. It is well known (and trivial) that if S is a right Noetherian ring then R is a right Noetherian ring. The converse is false in general as can be seen by taking $R = \begin{pmatrix} Q & Q \\ Q & Q \end{pmatrix}$ and $S = \begin{pmatrix} Q & Q \\ 0 & Z \end{pmatrix}$. However, P. M. Eakin [3] (and later M. Nagata [10]) showed that the converse holds if R is assumed to be a commutative ring. D. Eisenbud [4] and J. E. Björk [1] have extended Eakin's theorem to some mildly noncommutative situations.

In this note, we provide two mildly noncommutative versions of Eakin's theorem. The first version answers a question raised by J. E. Björk [8]. The second version improves upon Eisenbud's generalization of Eakin's theorem and, when applied to group algebras, yields a theorem of Clifford [2, p. 343].

As usual, all rings, subrings and modules are assumed to be unitary. Recall that a module M_R is *finite dimensional* if it does not contain any infinite direct sum of nonzero submodules. It is known [7, p. 216] that if M_R is finite dimensional then there exists a nonnegative integer n such that any direct sum of nonzero submodules of M contains at most n terms. The least such integer is called the *uniform dimension* of M and is denoted as $d(M)$ or $d_R(M)$.

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Our proofs of our main results have a certain strategy in common. In each case, we start with a Noetherian object and want to show that some related object is Noetherian. Due to the Noetherian induction, this presents no problem in lots of situations. Once the dubious cases are singled out, we show that the following lemma or the idea underlying its proof is applicable.

1. Lemma. *Let M be a finite dimensional module. If M/N is Noetherian for every essential submodule N of M then M is Noetherian.*

Proof. Let $\{M_n : n \geq 1\}$ be an ascending chain of submodules of M. Since $d(M_n)$ is a nondecreasing sequence of nonnegative integers which is bounded above by $d(M)$, it is eventually constant; say, $d(M_n) = d(M_m)$ for all $n \geq m$. We can use the finite dimensionality of M (or Zorn's lemma) to obtain a submodule K of M such that the sum $K + M_m$ is direct and essential in M. For $n \geq m$, we have $(K \cap M_n) \oplus M_m \subseteq M_n$ and $d(M_n) = d(M_m)$. It follows that $K \cap M_n = (0)$. Since the chain $\{K \oplus M_n : n \geq 1\}$ stops, so does $\{M_n : n \geq 1\}$. □

The following lemma is essentially Eakin's theorem stated for modules rather than rings. Its proof illustrates the strategy indicated above.

2. Lemma. *Let S be a subring of a commutative Noetherian ring R. Then any R-module which is finitely generated over S is Noetherian over S.*

Proof. Let M be an R-module which is finitely generated over S. Assume that every proper R-homomorphic image of M is Noetherian over S. We proceed to show that M itself is Noetherian over S.

Among the nonzero R-submodules of M which are finitely generated over S, choose L with largest possible $\operatorname{ann}_R L$. Since M/L is Noetherian over S, it suffices to show that L is Noetherian over $T = S/\operatorname{ann}_S L$. Note that T is a domain. For, if x, y are nonzero elements of T such that $xy = 0$ then Lx is a nonzero R-submodule of M which is finitely generated over S and annihilated by y, contradicting our choice of L.

Consider the T-torsion submodule N of L. It is in fact an R-submodule of L. Further, it is a proper submodule since L is finitely generated and faithful over the domain T. If $N \neq (0)$ then L/N is a (torsion-free and so) faithful Noetherian T-module. It follows that T is a Noetherian ring [9, p. 53], and so L is Noetherian over T.

It remains to treat the case when $N = (0)$. In this case, L is finitely generated and torsion-free over T; so, it is finite dimensional over T. If

K is any essential T-submodule of L then L/K is finitely generated and torsion so unfaithful over T. This yields a nonzero $t \in T$ such that $Lt \subseteq K$. Since Lt is a R-submodule of L, L/Lt and so L/K is Noetherian over T. Lemma 1 shows that L is Noetherian over T. Now a routine Noetherian induction proves the lemma. □

An alternate proof of Lemma 2 can be obtained by using the main result of [5]. We now prove a version of Eakin's theorem which answers a question of J. E. Björk [8, p. 376].

3. **Theorem.** *Let S be a commutative subring of a right Noetherian ring R such that R is finitely generated as a right S-module. Then S is a Noetherian ring.*

Proof. Clearly, it suffices to show that R is Noetherian as a right S-module. In view of the Noetherian induction, we may assume without loss that for every two-sided ideal I of R, R/I is Noetherian as a right S-module.

Suppose R is not a prime ring. Then there exists a finite number of prime ideals $P_1, \cdots, P_k$ of R such that $\bigcap\{P_i: 1 \leq i \leq k\}$ is the prime radical $P(R)$ of R. By Levitski's theorem, $\{P(R)\}^l = (0)$ for some positive integer l. It is clear that $\{P(R)\}^{i-1}/\{P(R)\}^i$, $1 \leq i \leq l$, is an R-homomorphic image of a finite direct sum of the right R-module $R/P(R)$. Thus, if $P(R) \neq (0)$ then R_S is Noetherian. If $P(R) = (0)$ then we must have $k > 1$ since R is not a prime ring. Then R embeds in $\Pi\{R/P_i: 1 \leq i \leq k\}$ as a right R-module which makes R_S Noetherian.

We now treat the (dubious) case when R is a prime ring. A result due to Procesi and Small [11] shows that End R_S satisfies a polynomial identity. Since R is isomorphic with a subring of End R_S, R is a prime P. I. ring. A theorem due to Formanek [6] now shows that there exists a $Z(R)$-monomorphism $R \hookrightarrow Z(R)^{(m)}$ for some positive integer m. Since the centre $Z(R)$ of R is a domain, it follows that R is a finite dimensional $Z(R)$-module.

Set $T = SZ(R)$. Let $\{M_n: n \geq 1\}$ be an ascending chain of T-submodules of R. There exists a positive integer h such that $d_{Z(R)}(M_n) = d_{Z(R)}(M_h)$ for all $n \geq h$. Choose a $Z(R)$-submodule K of R such that the sum $K + M_h$ is direct and essential in $R_{Z(R)}$. Then $K \cap M_n = (0)$ for all $n \geq h$ since $d_{Z(R)}(M_n) = d_{Z(R)}(M_h) < \infty$. Using the $Z(R)$-embedding $R \to Z(R)^{(m)}$, we see that $R/(K \oplus M_h)$ is unfaithful over $Z(R)$. This provides a nonzero $d \in Z(R)$ such that $Rd \subseteq K \oplus M_h$. It is easily seen that $M_n \cap$

$(Rd + M_h) = M_h$ for all $n \geq h$. So, $M_n/M_h \cong (Rd + M_n)/(Rd + M_h)$ as T-modules. Since R/Rd is Noetherian over S and so over T, it follows that the chain $\{M_n : n \geq 1\}$ stops. Thus R_T is Noetherian which makes T a Noetherian ring. By Lemma 2, R_S is a Noetherian module. □

The following theorem improves upon Eisenbud's generalization of of Eakin's theorem.

4. Theorem. *Let S be a subring of a ring R. Assume that there exists a finite subset $\{u_1, \cdots, u_m\}$ of R such that $u_iS = Su_i$ for $1 \leq i \leq m$ and $R = \Sigma\{u_iS: 1 \leq i \leq m\}$. Then a right R-module is Noetherian (resp. has a composition series) if and only if it is Noetherian (resp. has a composition series) as a right S-module. Every simple right R-module is finite dimensional and semisimple as a right S-module.*

Proof. After a trivial adjustment if necessary, we may assume that $u_1 = 1$.

Let M be a Noetherian right R-module such that every proper R-homomorphic image of M is Noetherian over S. If possible, let M contain an infinite direct sum $\bigoplus\{w_nS: n \geq 1\}$ of nonzero S-submodules. Using Zorn's lemma, we get a S-submodule W of M such that the sum $W + \Sigma\{w_nS: n \geq 1\}$ is direct and essential in M_S. We *claim* that, for each k $(1 \leq k \leq m)$, we have $d_k \in S$ and $n_k \in \mathbf{Z}^+$ such that $w_1d_k \neq 0$ and $w_1d_ku_i \in W + \Sigma\{w_nS: 1 \leq n \leq n_k\}$ for $i = 1, \cdots, k$. The claim is trivial for $k = 1$ since $u_1 = 1$. Inductively assume that the claim is valid for k, $1 \leq k < m$. If $w_1d_ku_{k+1} = 0$ then it suffices to take $d_k = d_{k+1}$ and $n_k = n_{k+1}$. If $w_1d_ku_{k+1} \neq 0$ then there exists $s \in S$ such that $0 \neq w_1d_ku_{k+1}s \in W + \Sigma\{w_nS: n \geq 1\}$. Since $u_iS = Su_i$ for $1 \leq i \leq m$, we have $s', s_1, \cdots, s_k$, in S such that

$$s'u_{k+1} = u_{k+1}s, \qquad s'u_i = u_is_i \quad \text{for } i = 1, \cdots, k.$$

Set $d_{k+1} = d_ks'$. Then $0 \neq w_1d_ku_{k+1}s = w_1d_{k+1}u_{k+1} \in W + \Sigma\{w_nS: n \geq 1\}$. Clearly, $w_1d_{k+1} \neq 0$. Further, we can choose $n_{k+1} \geq n_k$ such that $w_1d_{k+1}u_{k+1} \in W + \Sigma\{w_nS: 1 \leq n \leq n_{k+1}\}$. Then, for $1 \leq i \leq k$, we have

$$w_1d_{k+1}u_i = w_1d_ku_is_i \in W + \sum\{w_nS: 1 \leq n \leq n_{k+1}\}.$$

This completes the induction on k and proves our claim. It follows that

$$(0) \neq w_1d_mR \subseteq W + \sum\{w_nS: 1 \leq n \leq n_m\}.$$

Since M/w_1d_mR is Noetherian over S so also is $M/[W + \Sigma\{w_nS: 1 \leq n \leq n_m\}]$

which is impossible since it contains an infinite direct sum of nonzero S-submodules. We are thus forced to conclude that M is a finite dimensional S-module.

Let N be an essential S-submodule of M. As in the above paragraph, we obtain a nonzero R-submodule of M contained in N. Thus M/N is a Noetherian S-module. By Lemma 1, M is a Noetherian S-module.

Now a routine Noetherian induction shows that any Noetherian right R-module remains Noetherian over S. The converse is trivial.

Let M be a simple right R-module. As seen above, M_S is Noetherian and so finite dimensional. Let $\{L_n : n \geq 1\}$ be a descending chain of S-submodules of M. There exists a positive integer h such that $d_S(L_n) = d_S(L_h)$ for all $n \geq h$. Also, there exists a S-submodule K of M such that the sum $K + L_h$ is direct and essential in M_S. Then, for all $n \geq h$, $K + L_n$ is direct and essential in M_S. Imitating the argument in the second paragraph of the proof and using the simplicity of M_R, it follows that $M = K \oplus L_n$ for all $n \geq h$. So, $L_n = L_h$ for all $n \geq h$. Thus M_S is Artinian as well. So, soc M_S is finite dimensional and essential in M_S. A repetition of the above argument yields $M = \operatorname{soc} M_S$. The remaining part is trivial. □

The following corollary is a slight generalization of Clifford's theorem [2, p. 343].

5. **Corollary.** *Let S be a subring of a commutative ring R such that R is finitely generated as a S-module and let H be a normal subgroup of finite index in a group G. Then any simple right $R(G)$-module is finite dimensional and semisimple as a $S(H)$-module.* □

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