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ON THE UNIFORM ERGODIC THEOREM. II

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ABSTRACT. Let $\{T_t\}$ be a strongly continuous semigroup of bounded linear operators on a Banach space X, satisfying $\lim_{t\to\infty}\|T_t\|/t = 0$. We prove the equivalence of the following conditions: (1) $t^{-1}\int_0^t T_r\,dr$ converges uniformly as $t \to \infty$. (2) The infinitesimal generator A has closed range. (3) $\lim_{\lambda\to 0^+}\lambda R_\lambda$ exists in the uniform operator topology.

In [6] the author has proved that for a bounded linear operator T on a Banach space X satisfying $\|T^n/n\| \to 0$, $N^{-1}\sum_{n=0}^{N-1}T^n$ converges uniformly if and only if $(I-T)X$ is closed. In this paper we obtain the analogous result for the continuous case, generalizing some results of Hille and Phillips [5, Theorem 18.8.4] without using the operational calculus devised by Hille. Abel convergence in the discrete case is treated in the appendix.

Theorem. *Let $\{T_t\}_{t\geq 0}$ be a strongly continuous semigroup of bounded linear operators with $T_0 = I$, satisfying $\|T_t/t\| \to 0$. Let A be the infinitesimal generator of $\{T_t\}$ and let $R_\lambda x = \int_0^\infty e^{-\lambda t}T_t x\,dt$ be the resolvent family $(\lambda > 0)$. Then the following conditions are equivalent:*

(1) *There exists a bounded linear operator E such that $\|t^{-1}\int_0^t T_s\,ds - E\| \xrightarrow[t\to\infty]{} 0$.*

(2) *A has closed range.*

(3) *$N^{-1}\sum_{n=0}^{N-1}R_1^n$ converges uniformly.*

(4) *There exists a projection E on $\{x: T_t x = x, \forall\, t > 0\}$ such that $\lim_{\lambda\to 0^+}\|\lambda R_\lambda - E\| = 0$.*

Lemma 1. *Under the assumptions of the theorem, $\lim_{n\to\infty}\|(\lambda R_\lambda)^n\|/n = 0$ for every $\lambda > 0$.*

Proof. If t is large, $\|T_t\| \leq \epsilon t$ and $w_0 = \lim_{t\to\infty}\log\|T_t\|/t \leq$

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$\lim t^{-1}(\log t + \log \epsilon) = 0$. In [2, Lemma VIII.1.12] it is shown that for $\lambda > 0 \geq w_0$,

$$R_\lambda^n x = (n-1)!^{-1} \int_0^\infty e^{-\lambda t} t^{n-1} T_t x \, dt.$$

By induction and integration by parts we have $(\lambda^n/n!) \int_0^\infty e^{-\lambda t} t^n \, dt = 1/\lambda$.

Fix $\epsilon > 0$. Then $\|T_t\| \leq \epsilon t$ for $t \geq t_0(\epsilon)$. Since $\|T_t x\|$ is continuous on $[0, t_0]$, it is bounded, and by the principle of uniform boundedness $\|T_t\| \leq K$ for $0 \leq t \leq t_0$. Thus

$$\|(\lambda R_\lambda)^n\|/n \leq \frac{\lambda^n}{n!} K \int_0^{t_0} e^{-\lambda t} t^{n-1} \, dt + \epsilon \frac{\lambda^n}{n!} \int_{t_0}^\infty e^{-\lambda t} t^{n-1} t \, dt \leq \frac{K}{n} + \frac{\epsilon}{\lambda}$$

and $\limsup_n \|(\lambda R_\lambda)^n\|/n \leq \epsilon/\lambda$. Let $\epsilon \to 0$ to conclude the proof.

Lemma 2. *If $\mu R_\mu x = x$ for some $\mu > 0$, then $T_t x = x$ for every $t > 0$.*

Proof. By the resolvent equation $(R_\lambda - R_\mu)x = (\mu - \lambda) R_\lambda R_\mu x$.

$$R_\lambda x - R_\mu x = R_\lambda \mu R_\mu x - \lambda R_\lambda R_\mu x = R_\lambda x - \lambda R_\lambda R_\mu x.$$

Thus

$$(I - \lambda R_\lambda) x = \mu (I - \lambda R_\lambda) R_\mu x = 0,$$

and $\lambda R_\lambda x = x$ for every $\lambda > 0$. The lemma follows from the inversion formula (11.7.2) in [5].

Remark. If $\sup_{t \geq 0} \|T_t\| \leq M$, we may assume $M = 1$. Then Lemma 1 is trivial, and Lemma 2 is proved by Falkowitz [3] without using the inversion formula.

Proof of the Theorem. We denote by $R(A)$ the range of A and by Y the closure of $R(A)$.

(2) $\Rightarrow$ (3). For $\lambda > 0$ $(\lambda I - A) R_\lambda = I$ and for $x \in D(A)$, $R_\lambda(\lambda I - A)x = x$ [2, Lemma VIII.1.7]. Hence $\lambda R_\lambda = I + A R_\lambda$ and $(\lambda R_\lambda - I) X \subset R(A)$. By (2) $Y = R(A)$, so that for $y \in Y$ there is an $x \in D(A)$ with $Ax = y$, and

$$x = R_\lambda(\lambda I - A)x = \lambda R_\lambda x - R_\lambda y \quad \text{or} \quad R_\lambda y = (\lambda R_\lambda - I)x.$$

Thus

$$y = (\lambda I - A) R_\lambda y = (\lambda I - A)(\lambda R_\lambda - I)x = (\lambda R_\lambda - I)(\lambda I - A)x$$

and $Y = (\lambda R_\lambda - I)X$. Thus by applying the uniform ergodic theorem of [6] to the operator λR_λ we obtain (3), since $\|(\lambda R_\lambda)^n\|/n \to 0$ by Lemma 1.

(2) $\Rightarrow$ (1). For $x \in D(A)$, $A T_t x = T_t A x$ and therefore $T_t Y \subset Y$. The generator A_1 of the restriction of $\{T_t\}$ to Y will be the restriction of A to $Y \cap D(A)$. It is shown above that (2) implies $Y = (I - \lambda R_\lambda)X$, and the uniform ergodic theorem shows that $I - \lambda R_\lambda$ is invertible on Y. If $A_1 y = 0$

for $y \in Y \cap D(A)$, then

$$y = R_\lambda(\lambda I - A)y = \lambda R_\lambda y \quad \text{or} \quad (I - \lambda R_\lambda)y = 0,$$

and $y = 0$. Thus A_1 is one-to-one. On Y we have, as above, $(I - \lambda R_\lambda)Y \subset R(A_1)$. But from (2) $\Rightarrow$ (3) we have

$$Y \supset R(A_1) \supset (I - \lambda R_\lambda)Y = (I - \lambda R_\lambda)X = Y = R(A),$$

and $R(A_1) = Y$, so A_1^{-1} is defined for all of Y. A_1 is closed, therefore A_1^{-1} is closed, and by the closed graph theorem A_1^{-1} is continuous. For $y \in Y = R(A_1)$ there is an $x \in D(A) \cap Y$ such that $Ax = y$ and $\|x\| \leq \|A_1^{-1}\| \, \|y\|$. By [2, Lemma VIII.1.7]

$$(T_t - I)x = \int_0^t T_r Ax \, dr = \int_0^t T_r y \, dr$$

and $\|t^{-1}\int_0^t T_r y \, dr\| \leq \|A_1^{-1}\|(\|T_t\| + 1)t^{-1}\|y\|$. Hence on Y we have uniform convergence to 0. But $X = Y \oplus \{x: R_1 x = x\}$ by the uniform ergodic theorem, Lemma 2 then yields that $X = Y \oplus \{x: T_t x = x, \ \forall \ t > 0\}$, and (1) follows immediately.

(3) $\Rightarrow$ (2). By the uniform ergodic theorem $(I - R_1)X$ is closed. In (2) $\Rightarrow$ (3) we have shown $(I - R_1)X = R(A)$ and $R(A)$ is closed.

(1) $\Rightarrow$ (2). It follows from (1) that $E^2 = E$, with $EX = \{x: T_t x = x, \ \forall t \geq 0\}$. As shown above, $T_t Y \subset Y$. For $x \in D(A)$, we have by [2, Lemma VIII.1.7] that $(T_t - I)x = \int_0^t T_r Ax \, dr \in Y$. Since $D(A)$ is dense, we have that $clm \bigcup_{t \geq 0}(I - T_t)X \subset Y$. On the other hand, for $x \in D(A)$, $Ax = \lim_{h \to 0} h^{-1}(T_h - I)x$, so that $Y \subset clm \bigcup_{t \geq 0}(I - T_t)X$ and equality holds. Hence by (1), $X = EX \oplus Y$ (see [2, VIII.7.2]). Thus by restricting ourselves to Y we have $Y = \overline{R(A_1)}$ (where A_1 is the generator of the restriction, with domain $D(A) \cap Y$). Thus we may and do assume $X = Y$, and $\|t^{-1}\int_0^t T_s \, ds\| \to 0$.

Let $x \in D(A)$ satisfy $Ax = 0$. Then $x = R_1(I - A)x = R_1 x$ and by Lemma 2 $T_t x = x$ for every $t \geq 0$. Since $X = Y$, $x = 0$ and A is one-to-one, with A^{-1} defined on $R(A)$. For fixed t large enough, $\|t^{-1}\int_0^t T_r dr\| < 1$ and $I - t^{-1}\int_0^t T_r dr$ is invertible, and so is $\int_0^t (T_r - I) dr$. For $y \in R(A)$ take $x \in D(A)$ with $Ax = y$. Then

$$\int_0^t (T_s - I)x \, ds = \int_0^t \left(\int_0^s T_r Ax \, dr\right) ds = \int_0^t \left(\int_0^s T_r y \, dr\right) ds$$

and

$$\|x\| = \|A^{-1}y\| \leq \left\| \left(\int_0^t (T_s - I) \, ds\right)^{-1} \right\| \left\| \int_0^t \left(\int_0^s T_r y \, dr\right) ds \right\| \leq K\|y\|.$$

Hence A^{-1} is continuous with dense domain (in $X = Y$). Since A is closed, A^{-1} is defined on all of Y, and $Y = R(A_1) \subset R(A)$, so that $R(A) = Y$ is closed.

(1) $\Rightarrow$ (4). We have just seen above that $X = EX \oplus R(A)$, with $EX = \{x: T_t x = x,\ t \geq 0\}$, so that we have to show only $\lambda R_\lambda \to 0$ uniformly on $Y = R(A) = R(A_1)$. Again we may and do assume $X = Y$, and A^{-1} is continuous. Let $0 < \lambda \leq \delta < 1/\|A^{-1}\|$. For $y \in Y$

$$\|\lambda R_\lambda y\| = \|\lambda R_\lambda A A^{-1} y\| \leq \|\lambda(\lambda R_\lambda - I)\|\,\|A^{-1}\|\,\|y\|.$$

Hence for $0 < \lambda \leq \delta$ we have,

$$\|\lambda R_\lambda\| \leq (1 + \|\lambda R_\lambda\|)\delta\|A^{-1}\|, \quad \text{or} \quad \|\lambda R_\lambda\| \leq \delta\|A^{-1}\|/(1 - \delta\|A^{-1}\|) \equiv M.$$

Now we have

$$\|\lambda R_\lambda\| \leq \|\lambda(\lambda R_\lambda - I)\|\,\|A^{-1}\| \leq \lambda(1 + M)\|A^{-1}\| \underset{\lambda \to 0^+}{\to} 0.$$

(4) $\Rightarrow$ (2). If $y = Ax$ for $x \in D(A)$, then $\lambda R_\lambda y = \lambda R_\lambda A x = \lambda(\lambda R_\lambda - I)x \underset{\lambda \to 0^+}{\to} 0$. Restricting ourselves to Y we have $\|\lambda R_\lambda\|_Y \to 0$. But on Y we have for $\lambda > 0$ small enough that $I - \lambda R_\lambda$ is invertible, hence

$$Y = (I - \lambda R_\lambda)Y \subset (I - \lambda R_\lambda)X = R(A)$$

(see (2) $\Rightarrow$ (1) for last equality) and $R(A) = Y$.

Remark. [5, Theorem 18.8.4] treats more general semigroups, but is only concerned with Abel convergence, and does not include the sufficiency of $R(A)$ being closed. The method presented here is different, not using the operational calculus and spectral theory used in [5].

Corollary 1. *Let $\{T_t\}$ be as above. The following two conditions are equivalent:*

(1) $t^{-1}\int_0^t T_r\,dr$ *converges uniformly as* $t \to \infty$.

(2) *There is a* $\delta > 0$ *such that* $\sup_{0<\lambda\leq\delta}\|R_\lambda y\| < \infty$ *for every* $y \in \overline{R(A)}$.

If $\|T_t\| \leq M$ *for* $t \geq 0$, *then the following (sufficient) condition is also equivalent to the previous two:*

(3) $\sup_{t \geq 0}\|\int_0^t T_r y\,dr\| < \infty$ *for every* $y \in \overline{R(A)}$.

Proof. (1) $\Rightarrow$ (2). By the Theorem $R(A)$ is closed. For $y \in R(A)$ we have $x \in D(A)$ with $Ax = y$, and $R_\lambda y = R_\lambda A x = -x + \lambda R_\lambda x$. The proof of (1) $\Rightarrow$ (4) in the Theorem shows that $\sup_{0<\lambda\leq\delta}\|\lambda R_\lambda\| < \infty$, and (2) follows. (In fact, $\sup_{0<\lambda<\infty}\|R_\lambda y\| < \infty$: by the computations of Lemma 1, $\sup_{\delta\leq\lambda}\|\lambda R_\lambda\| < \infty$.)

(2) $\Rightarrow$ (1). We look at the semigroup restricted to $Y = \overline{R(A)}$, with generator A_1. By the principle of uniform boundedness, $\sup_{0<\lambda\le\delta}\|R_\lambda\|_Y < \infty$ and therefore $\lim_{\lambda\to 0}\|\lambda R_\lambda\|_Y = 0$. By the Theorem, $Y = R(A_1) \subset R(A)$. Hence $R(A)$ is closed and (1) follows from the Theorem.

(3) $\Rightarrow$ (1) is proved similarly.

If $\|T_t\| \le M$ for every $t \ge 0$ and we assume (1), then for $y = Ax$ $(x \in D(A))$ we have, by [2, Lemma VIII.1.7],

$$\left\|\int_0^t T_r y\,dr\right\| = \left\|\int_0^t T_r Ax\,dr\right\| = \|(T_t - I)x\| \le (M+1)\|x\|.$$

Since $R(A)$ is closed, (3) follows.

Corollary 2. *Assume $\|T_t\| \le M$ for every $t \ge 0$. Let ν be a probability measure on $[0, \infty)$, $\nu(\{0\}) < 1$, and define $Ux = \int T_t x\,d\nu$. If $N^{-1}\sum_{n=0}^{N-1} U^n$ converges uniformly, then $t^{-1}\int_0^t T_r\,dr$ converges uniformly (as $t \to \infty$).*

Proof. Since $|||x||| = \sup_{t\ge 0}\|T_t x\|$ is an equivalent norm for X such that $|||T_t||| \le 1$, we may and do assume $\|T_t\| \le 1$.

Note that U is well defined in the strong operator topology, by [5, Theorem 3.7.4].

By the discrete uniform ergodic theorem $X = (I-U)X \oplus \{x: Ux = x\}$; since T_t commutes with U, both subspaces are invariant under $\{T_t\}$, and we can restrict ourselves to each one. Hence it suffices to treat two cases: (1) $I - U$ is invertible. (2) $U = I$.

Case 1. Fix $\epsilon > 0$. Let $\beta > 0$ be such that $\nu([0, \beta]) > 1 - \epsilon$. Define $\nu_\epsilon(C) = \nu(C \cap [0, \beta])/\nu([0, \beta])$. Then $\|\nu - \nu_\epsilon\| \le 1/(1-\epsilon) - 1 + \epsilon$. Define $U_\epsilon x = \int T_t x\,d\nu_\epsilon$. Then

$$\|(I-U) - (I-U_\epsilon)\| = \|U - U_\epsilon\| \le \|\nu - \nu_\epsilon\| \underset{\epsilon\to 0}{\longrightarrow} 0$$

and $I - U_\epsilon$ is also invertible for ϵ small enough. Thus we may and do assume that ν is supported on a finite interval $[0, \beta]$.

Let A be the infinitesimal generator of $\{T_t\}$. Since $D(A)$ is dense, for $y \in X$ there is a sequence $x_n \in D(A)$ with $x_n \to (I-U)^{-1}y$. Hence

$$y = \lim_{n\to\infty}(I-U)x_n = \lim_{n\to\infty}\int (I - T_t)x_n\,d\nu = -\lim_{n\to\infty}\int\left(\int_0^t T_r A x_n\,dr\right)d\nu,$$

which shows that $y \in \overline{R(A)}$ (since $\overline{R(A)}$ is invariant under $\{T_t\}$). Let $z_n \in D(A)$ satisfy $Az_n \to y$. Then

$$\|(I-U)z_n-(I-U)z_m\| = \left\| \int \left\{ \int_0^t T_r(Az_n - Az_m)\,dr \right\} d\nu \right\|$$
$$\leq \int \left\{ \int_0^t \|Az_n - Az_m\|\,dr \right\} d\nu$$
$$= \|Az_n - Az_m\| \int t\,d\nu \leq \beta \|Az_n - Az_m\|.$$

Thus $\{(I-U)z_n\}$ is a Cauchy sequence, and $z_n = (I-U)^{-1}(I-U)z_n$ converges (strongly), say to z. Then $z_n \to z$ and $Az_n \to y$, and since A is a closed operator, $z \in D(A)$ and $y = Az$. Thus $X = R(A)$ and by the theorem $\|t^{-1}\int_0^t T_r\,dr\| \to 0$.

Case 2. We show first that $U = I$ implies $T_{t_0} = I$ for some $t_0 > 0$. If, for $t_0 > 0$, $\alpha = \nu(\{t_0\}) > 0$, then for every $x \in X$ we have $x = Ux = \alpha T_{t_0} x + (1-\alpha)\int T_t x\,d\nu_1$, where $\nu_1 = (\nu - \alpha\delta_{t_0})/(1-\alpha)$ with δ_{t_0} the Dirac measure at t_0. By [3, Lemma 1], $T_{t_0} x = x$, and thus $T_{t_0} = I$. Assume now that $\nu(\{t\}) = 0$ for $t > 0$. Let $F(t) = \nu((-\infty, t])$ be the distribution of ν. Then $F(t)$ is continuous at $t > 0$, and since $F(0) = \nu(\{0\}) < 1$, there is a $t' > 0$ with $0 < F(t') = a < 1$. Let $t_0 = \sup\{t: F(t) = a\}$, and $F(t_0) = a$ by continuity. Take $t_0 < t_n \downarrow t_0$, and let $a_n = \nu([t_0, t_n])$. Define (since $1 > a_n > 0$)

$$\nu_n(C) = \nu(C \cap [t_0, t_n])/a_n \quad \text{and} \quad \mu_n(C) = \{\nu(C) - a_n\nu_n(C)\}/(1-a_n).$$

Then ν_n and μ_n are nonzero probability measures with $a_n\nu_n + (1-a_n)\mu_n = \nu$. By [3, Lemma 1] we also have that $\int T_t x\,d\nu_n = x$ for every $x \in X$, so that $\int T_t\,d\nu_n = I$. Fix $x \in X$, $x^* \in X^*$. Then $\langle x^*, T_t x\rangle$ is continuous on $[t_0, t_n]$ and there is a point $t_0 \leq s_n \leq t_n$ such that $\langle x^*, T_{s_n} x\rangle = \int \langle x^*, T_t x\rangle\,d\nu_n = \langle x^*, x\rangle$. Since also $\lim s_n = t_0$, the continuity of $\{T_t\}$ yields $\langle x^*, T_{t_0} x\rangle = \langle x^*, x\rangle$. Hence $T_{t_0} = I$. To finish the proof in Case 2, denote $n = [t/t_0]$. Then for $t > t_0$ we have (with $\alpha = t/t_0 - n$)

$$t^{-1}\int_0^t T_r\,dr = t^{-1}\int_0^{nt_0} T_r\,dr + t^{-1}\int_{nt_0}^t T_r\,dr$$
$$= t^{-1}n\int_0^{t_0} T_r\,dr + t^{-1}T_{nt_0}\int_0^{\alpha t_0} T_r\,dr \to t_0^{-1}\int_0^{t_0} T_r\,dr.$$

Remarks. (1) The corollary can be applied if $I - U - Q$ is invertible for some compact operator Q (see [6]). (2) In the discrete case $\{T^n\}$ treated in [6] there is a requirement $\sum_{i=1}^k a_i = 1$. It can be removed by an approxima-

tion argument, as in the proof of Corollary 2. (Also see the example in the appendix.)

Appendix. We show here that if $\|T^n/n\| \to 0$, then Abel and Cesàro convergence of $\{T^n\}$ are equivalent in the uniform operator topology. Hille [4] proved it for the case $\sup\|T^n\| < \infty$, using a general method which does not apply in the case that T is not power-bounded. (An extension of Hille's result in the strong operator topology is a consequence of [1, Theorem 1].)

Proposition. *Let T be a linear operator on X with $\|T^n/n\| \to 0$. Then the following conditions are equivalent:*

(1) $N^{-1}\sum_{n=0}^{N-1} T^n$ *converges in the uniform operator topology.*

(2) $\lim_{r\to 1^-} \|(1-r)\sum_{n=0}^{\infty} r^n T^n - E\| = 0$ *for some operator E.*

(3) $(I-T)X$ *is closed.*

Proof. Let $Y = \overline{(I-T)X}$, which is invariant under T, and let S on Y be the restriction of T.

(1) $\Rightarrow$ (2). Since this follows from Hille's general results [4], we only sketch a simple proof for the present situation. From (1) we have $X = Y \oplus \{x: Tx = x\}$, and we have only to show $\|(1-r)\sum_{n=0}^{\infty} r^n S^n\| \to 0$. But by the uniform ergodic theorem, $I - S$ is invertible on Y. Thus, for $0 < r < 1$,

$$\left\|(1-r)\sum_{n=0}^{\infty} r^n S^n\right\| = \left\|(1-r)\sum_{n=0}^{\infty} r^n S^n (I-S)(I-S)^{-1}\right\|$$

$$\leq \left\{\left\|(1-r)\sum_{n=1}^{\infty}(r^n - r^{n-1})S^n\right\| + (1-r)\right\}\|(I-S)^{-1}\|$$

$$\leq (1-r)\|(I-S)^{-1}\|\left\{1 + (1-r)\sum_{n=1}^{\infty} r^{n-1}\|S^n\|\right\}.$$

Fix $\epsilon > 0$. For $n > k$, $\|S^n\|/n < \epsilon$ and

$$(1-r)^2\sum_{n=1}^{\infty} r^{n-1}\|S^n\| \leq (1-r)^2\sum_{n=1}^{k} r^{n-1}\|S^n\| + \epsilon(1-r)^2\sum_{n=1}^{\infty} r^{n-1}n$$

$$= (1-r)^2\sum_{n=1}^{k} r^{n-1}\|S^n\| + \epsilon \underset{r\to 1^-}{\longrightarrow} \epsilon.$$

Let $\epsilon \to 0$ to conclude (2).

(2) $\Rightarrow$ (3). We shall show $(I-S)Y = Y$, which implies (3). Hence we restrict ourselves to Y. If $y = (I-T)x$, then

$$(1-r)\sum_{n=0}^{\infty} r^n S^n y = (1-r)\sum_{n=0}^{\infty} r^n (I-T)T^n x$$

$$= (1-r)\sum_{n=0}^{\infty} r^n T^n x - (1-r)r^{-1}\sum_{n=0}^{\infty} r^n T^n x + (1-r)r^{-1}x$$

$$\to Ex - Ex + 0 = 0.$$

Hence $\|(1-r)\sum_{n=0}^{\infty} r^n S^n\| \underset{r\to 1^-}{\to} 0$. Thus for fixed $0 < r < 1$, close enough to 1, $I - (1-r)\sum_{n=0}^{\infty} r^n S^n$ is invertible. Since $\sum_{n=0}^{\infty} r^n S^n$ is defined in the uniform operator topology (by virtue of $\|T^n\|/n \to 0$), we have that

$$(1-r)\sum_{n=1}^{\infty} r^n \left(\sum_{i=0}^{n-1} S^i\right) = (1-r)\sum_{i=0}^{\infty}\left(\sum_{n=i+1}^{\infty} r^n\right) S^i = \sum_{i=0}^{\infty} r^{i+1} S^i$$

is also defined in the uniform operator topology. But

$$I - (1-r)\sum_{n=0}^{\infty} r^n S^n = (1-r)\sum_{n=0}^{\infty} r^n (I - S^n) = (I-S)(1-r)\sum_{n=1}^{\infty} r^n \left(\sum_{i=0}^{n-1} S^i\right)$$

and $(I-S)Y = Y$ by invertibility of the left-hand side.

(3) $\Rightarrow$ (1) is proved in [6].

Example. We exhibit an operator T satisfying $\|T\| \leq 1$, $(I-T)X = X$ and $(I-T^2)X$ is not closed. Since $\|N^{-1}\sum_{n=0}^{N-1} T^n\| \to 0$, T satisfies the conditions of [6, Corollary 2] (with $Q = 0$), but is not quasi-compact (since this implies T^k are all quasi-compact).

Let $X = l_2$, $0 > \lambda_1 > \lambda_2 > \cdots > \lambda_n > \downarrow -1$. Define $Tx = \{\lambda_i x_i\}$ where $x = \{x_i\}$. Then $(I-T)^{-1}x = \{x_i/(1-\lambda_i)\}$, $(I-T^2)X$ contains all sequences with finitely many nonzero terms, and is dense. But $-1 \in \sigma(T)$ and $I+T$ is not invertible, but $I+T$ is one-to-one. Hence,

$$(I-T^2)X \subset (I+T)X \neq X = \overline{(I-T^2)X}.$$

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