CHAIN CONDITIONS ON SYMMETRIC ELEMENTS

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ABSTRACT. Recently Britten has proven an analog of Goldie's theorem for the Jordan ring $S$ of symmetric elements in a ring with involution of characteristic not 2. In this paper we first extend Britten's theorem to the situation where $R$ is an arbitrary ring and the Jordan ring is only an ample subring of the symmetric elements. We apply this result to show that if $S$ has ACC on quadratic ideals, then the (Jordan) nil radical of $S$ is nilpotent.

Recently Britten [1] has proven an analog of Goldie's theorem for the Jordan ring of symmetric elements of a ring with involution. More specifically, he has shown that if $2R = R$ and the symmetric elements $S$ of $R$ are a prime Jordan ring with either ascending or descending chain condition on quadratic ideals, then $S \cong S_1$, the symmetric elements of a $*$-prime ring $R_1$, and $R_1$ has a ring of quotients which is $*$-simple Artinian.

In this paper, we first extend Britten's theorem to the situation where $R$ is an arbitrary ring and the Jordan ring is only an ample subring of the symmetric elements. The proof given here is considerably shorter than Britten's; the simplification comes from using an idea of Lanski [5] on direct sums of right ideals.

As an application, we show that if $S$ satisfies the ascending chain condition on quadratic ideals, then the nil radical of $S$ is nilpotent. This question remains open for general Jordan algebras.

Before beginning, we need some definitions. $R$ will always denote an associative ring, and $*$ an involution on $R$ (that is, an anti-automorphism of $R$ of period 2). We let $S_R = \{x \in R| x^* = x\}$ denote the symmetric elements of $R$, $T_R = \{x + x^*| x \in R\}$ the traces, and $N_R = \{xx^*| x \in R\}$ the norms. An ideal $I$ of $R$ is called a $*$-ideal if $I^* = I$; thus, $R$ is $*$-prime if the product of two nonzero $*$-ideals is nonzero, and $*$-simple if there are no
proper \(*\)-ideals. We say \(R\) is semiprime if \(R\) contains no nilpotent ideals; it is easy to see that a \(*\)-prime ring is semiprime.

In what follows, \(J\) will denote a special quadratic Jordan ring; that is, \(J\) will be an additive subgroup of \(R\) closed under the quadratic operator \(xU_y = yxy\) and the binary composition \(x^2\). A Jordan subring \(J\) of \(S_R\) is called ample if it contains all norms and traces and if \(x|x^* \subseteq J\), for all \(x \in R\). An additive subgroup \(I \subseteq J\) is a quadratic ideal of \(J\) if \(IU_I \subseteq I\).

We abbreviate the ascending (descending) chain condition by ACC (DCC).

Recall that \(R\) is a right Goldie ring if \(R\) has ACC on right annihilators and \(R\) has no infinite direct sums of right ideals.

We begin with an easy lemma. Part (ii) is part of Britten’s Lemma A [1].

**Lemma 1.** Let \(R\) be a semiprime ring with \(*\), and \(J\) an ample Jordan subring of \(S\). Then:

(i) If \(a \in J\) with \(a^\* a = 0\), then \(a = 0\).

(ii) If \(I\) is a right (left) ideal of \(R\) with \(I \cap J = 0\), then \(Q = \{x + x^* | x \in I\}\) is a quadratic ideal of \(J\).

**Proof.** (i) Since \(a \in J\) and \(J\) is ample, \(xax^* \in J\) for all \(x \in R\). Thus, \(axax^*a = 0\). On the other hand, \(x + x^* \in J\) so \(a(x + x^*)a = 0\). This gives \(axa = -ax^*a\), and so \(axaxa = 0\), all \(x \in R\). But then \((ax)^3 = 0\); that is, \(aR\) is a nil right ideal of \(R\) of bounded index. By a theorem of Levitzki [3, p. 1], \(R\) has a nilpotent ideal (a contradiction), unless \(a = 0\).

(ii) Say that \(I\) is a right ideal, and choose \(a \in I\). Then \(aJa^* \subseteq I \cap J = 0\). Thus, if \(s \in J\), \(a*sa \in J\) and \(a*saJa*sa = 0\). By (i), \(a*sa = 0\). Since \(s\) was arbitrary in \(J\), \(a*Ja = 0\). It now follows that \(JU_{a+a^*} \subseteq Q\), for all \(a \in I\).

The first theorem extends Theorem 1 of Britten.

**Theorem 1.** Let \(R\) be a \(*\)-prime ring and \(J\) ample. If \(J\) has no infinite direct sums of quadratic ideals, then \(R\) has no infinite direct sums of right or left ideals.

**Proof.** Assume that the result is false, that is, \(J\) has no infinite direct sums of quadratic ideals but, in \(R\), \(\mathcal{M} = \{T_i\}\) is an infinite collection of right ideals whose sum is direct.

We claim that we may assume (by replacing \(\mathcal{M}\) by an appropriate subset) that any finite sum of ideals in \(\mathcal{M}\) intersects \(J\) only in \((0)\). For, let \(\mathcal{M}_n = \{T_{n}, T_{n+1}, \ldots\}\). Now, if \(\mathcal{M} = \mathcal{M}_1\) fails, there exists a finite sum \(S_1 = T_1 + T_2 + \cdots + T_{n_1-1}\) such that \(S_1 \cap J \neq (0)\). Then try \(\mathcal{M}_{n_1} = \{T_{n_1}, \ldots\}\). If \(\mathcal{M}_{n_1}\) fails, there must be a finite sum \(S_2 = T_{n_1} + \cdots + T_{n_2-1}\) with...
$S_n \cap J \neq (0)$. Repeat the argument: if $\mathbb{M}_{n_k}$ fails, there is some $S_{k+1} \neq S_{n_k} + \cdots + S_{n_k+1} - 1$ with $S_{k+1} \cap J \neq (0)$. But if $S_k$ exists for all $k$, then the set $\{S_k \cap J\}$ is an infinite set of quadratic ideals of $J$ which form a direct sum, a contradiction. Thus, for some $k$, it must be that in $\mathbb{M}_{n_k}$, no finite sum of right ideals intersects $J$ nontrivially. $\mathbb{M}_{n_k}$ is the desired subset.

Now for each $T_i \in M$, consider $Q_i = \{x + x^* : x \in T_i\}$. By Lemma 1, $Q_i$ is a quadratic ideal of $J$. Since $J$ has no infinite direct sums of quadratic ideals, there is a finite set $\{Q_1, \ldots, Q_n\}$ which is dependent: for some $q_i \in Q_i$, not all zero, $\Sigma q_i = 0$. Write $q_i = x_i + x_i^*$ for $x_i \in T_i$ and let $a = \Sigma x_i \in \Sigma_{i=1}^n T_i$. Since there is no infinite direct sum in $\{Q_{n+1}, \ldots, Q_m\}$, there must be another finite dependent set $\{Q_{n+1}, \ldots, Q_m\}$. Say that $\Sigma p_j = 0$ for not all $p_j \in Q_j$ zero. Let $p_j = y_j + y_j^*$ and $b = \Sigma y_j \in \Sigma_{n+1}^m T_j$. Then $a + a^* = b + b^* = 0$; thus $a$ and $b$ are skew and are nonzero by independence of the $T_i$.

Now let $r \in R$. Then $arb + br^*a \in (\Sigma T_i + \Sigma T_j) \cap J = (0)$, and so $arb = -br^*a$. But then $arb \in (\Sigma T_i) \cap (\Sigma T_j) = (0)$ by the directness of $\{T_k\}$. Thus, $arb = 0$, all $r \in R$. But then $(RaR)(RbR) = 0$, and $RaR$ and $RbR$ are $*$-ideals of $R$ since $a$ and $b$ are skew. By the $*$-primeness of $R$, $a = 0$, or $b = 0$, a contradiction. Thus, $R$ cannot have an infinite direct sum of right ideals.

Now if $U$ is a subset of $R$, we write $\text{ann}_R U = \{x \in R \mid Ux = 0\}$, the right annihilator of $U$. Similarly, $\text{ann}_l U$ denotes the left annihilator of $U$. If $A_1 \subset A_2 \subset \cdots$ is a properly ascending chain of right annihilators in $R$, let $B_i = \text{ann}_{l} A_i$. Then it is easy to verify that the $B_i$ are a properly descending chain of left annihilators and that $A_i = \text{ann}_{l} B_i$.

The next lemma is also part of Britten's Lemma A.

**Lemma 2.** Let $R$ be $*$-prime. Then if $C$ is the right (left) annihilator of a nonzero ideal of $R$, $C \cap C^* = 0$.

**Proof.** Let $I$ be the ideal of $R$, so that $IC = 0$. Then $Cl$ is a right ideal of $R$ with $(C)I = 0$, and so $CI = 0$ since $R$ is semiprime. Applying $*$, we have $I^*C^* = 0$ and thus $(I + I^*)(C \cap C^*) = 0$. Since $R(C \cap C^*)R$ is a $*$-ideal of $R$, we must have $C \cap C^* = 0$ since $R$ is $*$-prime.

We are now able to extend Britten's Theorem 4. The argument is a simplification of his argument.

**Theorem 2.** Let $R$ be $*$-prime and $J$ ample. If $J$ has ACC or DCC on quadratic ideals, then $R$ is a Goldie ring.

**Proof.** If $J$ has ACC or DCC on quadratic ideals, then it is easy to see
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that $J$ has no infinite direct sums of quadratic ideals. Thus, by Theorem 1, it suffices to show that $R$ satisfies ACC on right annihilators.

We first treat the case when $J$ has ACC on quadratic ideals. If $R$ is not Goldie, let $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ be a proper ascending chain of right annihilators in $R$, and let $B_i$ be the corresponding descending chain of left annihilators.

First assume that $A_i \cap J \neq 0$ for some $i$. Since the $A_i$ are ascending, we may as well assume that $A_i \cap J \neq 0$ for all $i$. Then $\{A_i \cap J\}$ is an ascending chain of quadratic ideals of $J$, so for some $m$, $A_m \cap J = A_{m+1} \cap J = \cdots$. We claim that this gives a contradiction. For, choose $a \in A_m \cap J$ and $y \in A_{m+1}$. Then $ya + ay^* \in A_{m+1} \cap J = A_m \cap J$, and so $0 = B_m(ya + ay^*) = B_mya$, since $a \in A_m = \text{ann}_R B_m$. Thus $(B_m A_{m+1})a = 0$. Since $B_mA_{m+1} \neq 0$, and $a \in J \subseteq S$, $a = 0$ by Lemma 2. Thus $A_m \cap J = 0$.

We may therefore assume that $A_i \cap J = 0$, for all $i$. For each $A_i$, let $Q_i$ be given as in Lemma 1. Since the $Q_i$ are ascending, for some $m$, $Q_m = Q_{m+1} = \cdots$. We claim that $A_{m+1} = A_m$. For, if not, choose $a \in A_{m+1} \setminus A_m$. Then $a + a^* \in Q_{m+1} = Q_m$, so $a + a^* = a_m + a_m^*$, for some $a_m \in A_m$. Thus, $a - a_m = -(a - a_m^*) \in A_{m+1} \setminus A_m$, and is skew. So we may assume that $a$ is skew. Now choose any $y \in A_{m+2}$. Then $ay^* + ya^* = ay^* - ya \in A_{m+2} \cap J = 0$, so $ay^* = ya$. But then $B_{m+1} ay^* = 0 = B_{m+1}ya$ since $a \in A_{m+1}$. That is, $(B_{m+1} A_{m+2})a = 0$. As above, this gives $a = 0$, a contradiction. Thus, $A_{m+1} = A_m$, which contradicts the $A_i$ being a proper chain.

The proof for DCC is very similar; if $B_i \cap J = 0$ for all $i$, get the $Q_i$'s as above. Then we can choose a skew in $B_{m+1}$, not in $B_{m+2}$. If $y \in B_m$, we again have $y^*a = ay$, so $a(B_{m+1} A_m) = 0$ and $a = 0$ as before. Thus, we may assume that $B_i \cap J \neq 0$, for some $i$. But the $\{B_i \cap J\}$ are a proper descending chain of quadratic ideals, so for some $m$, $B_m \cap J = B_{m+1} \cap J = \cdots$ and we may assume $B_m \cap J \neq 0$. But then, choosing $a \in B_m \cap J$ and $y \in B_m$, we see that $a(B_{m} A_{m+1}) = 0$, so $a = 0$.

We are now able to extend Britten's main theorem:

**Corollary 1.** Let $J$ be an ample subring of $S_R$, where $R$ is a ring with *. If $J$ is prime and has ACC or DCC on quadratic ideals, then $J$ is isomorphic to an ample Jordan subring $J_1$ of a *-prime Goldie ring $R_1$.

**Proof.** Let $P(R)$ denote the prime radical of $R$. Then by the same proof as [2, Lemma 1], $P(R) \cap J \subseteq P(J)$, the prime radical of $J$. But since $J$ is prime, $P(J) = 0$; thus, $P(R) \cap J = 0$. Let $R_1 = R/P(R)$; $R_1$ is semiprime. Now $J_1$, the image of $J$ in $R_1$, is isomorphic to $J$ since $P(R) \cap J = 0$. Also $J_1$
is ample since \(J\) is \((R_1\) has an induced involution since \(P(R)\) * = \(P(R)\)).

It remains only to show that \(R_1\) is actually *-prime (since then \(R_1\) Goldie follows from Theorem 2). Say \(AB = 0\), where \(A, B\) are \(\neq 0\) *-ideals of \(R\). Then \(A \cap J \neq 0\) (since if \(A \cap J = 0\), \(a + a^* = 0 = a^*a = a^2\), for all \(a \in A\), and \(R\) would contain a nilpotent ideal). Similarly, \(B \cap J \neq 0\). But \(A \cap J\) and \(B \cap J\) are ideals of \(J\), and \((A \cap J)U_B \cap J = 0\), contradicting \(J\) being prime.

Since \(R_1\) is a *-prime Goldie ring, by Goldie's theorem \(R_1\) has a ring of quotients \(A\) which is semisimple Artinian. It is not difficult to see that in fact \(A\) is *-simple (see Britten's Theorem 5). We can therefore show that \(J\) has a Jordan ring of quotients.

Corollary 2. Let \(J, R, I, J, R_1\), and \(R_1\) be as in Corollary 1. Then \(J\) has a Jordan ring of quotients \(Q(J)\). If \(A\) is the associative ring of quotients for \(R_1\), then \(Q(J) \cong Q(J_1)\) is an ample subring of \(S_A\), and \(Q(S_{R_1}) = S_A\). When \(2R = R\), \(Q(J)\) is a simple Jordan ring.

Proof. Using Corollary 1, we may work entirely in \(R_1\). We claim that every regular element of \(J\) is regular in \(R_1\). For let \(r \in J\), regular, and say that \(rx = 0\) for some \(x \in R_1\). Then \((x + x^*)U_x = 0\), so \(x + x^* = 0\). Then \(x = -x^*,\) so \(x^2 = -xx^* \in J\). Since \(x^2U_x = 0\), \(x^2 = 0\). But this says that the right annihilator of \(r\) is nil of index 2, and so \(R_1\) contains a nilpotent ideal (a contradiction). Thus, \(x = 0\). Similarly, \(xr = 0\) implies \(x = 0\).

We can now apply [7, Theorem 4.5], which states exactly that \(Q(S_R) = S_A\), and \(Q(J)\) is ample whenever \(R\) has a ring of quotients \(A\) and every element of \(J\) is regular in \(R\).

The fact that when \(2R = R\), \(Q(J) \cong S_A\) is simple follows from a theorem of Herstein [3, Theorem 2.6].

Having extended Britten's results to arbitrary characteristic, we now apply them to study the nil radical of \(S\). Let \(N(J)\) denote the nil radical of the Jordan ring \(J\). As in Corollary 1, \(P(R)\) will be the (associative) prime radical of \(R\), and \(P(J)\) the (Jordan) prime radical of \(J\). We work with \(J\), rather than \(S\), since the symmetric elements might not be preserved under homomorphic images.

Lemma 3. Let \(R\) be a *-prime ring in which \(J\) has either ACC or DCC on quadratic ideals. Then \(N(J) = 0\).

Proof. Since \(J\) is ample and \(N(J)\) is an ideal of \(J\), \(N(J)\) is a core ideal
of $S$ (for details see [6, p. 387]). Thus, if $N(J) \neq 0$, there exists a nonzero associative *-ideal $B$ of $R$ such that the core $K_0(B) \subseteq N(J)$, where $K_0(B) = \{b + b^* + \sum b_i b^*_i | b, b_i \in B\}$ [6, Corollary, p. 387]. We claim that $B$ must contain a nonzero nil right ideal. If $K_0(B) = 0$, this is trivial; for $b + b^* = 0$, or $b^* = -b$, for all $b \in B$. But then also $bb^* = -b^2 = 0$, and so $B$ itself is nil. We may therefore assume that $K_0(B) \neq 0$. Choose $x \in K_0(B)$, $x \neq 0$. Then $x^n = 0$, $x^{n-1} \neq 0$ for some $n$, so letting $y = x^{n-1}$ we have $y^2 = 0$, $y \neq 0$. Now for any $b \in B$, $yb + b^*y \in K_0(B) \subseteq N(J)$, and so for some $k$, $(yb + b^*y)^k = 0$. Multiplying on the right by $yb$, we see $(yb)^{k+1} = 0$; that is, $yB$ is nil.

By Theorem 2, $R$ is a Goldie ring. By Lanski's theorem [4], any nil subring of $R$ is nilpotent. Thus, $B$ contains a nilpotent ideal. But $B$ is an ideal in a semiprime ring, so is itself a semiprime ring, a contradiction. Thus, $N(J) = 0$.

**Theorem 3.** Let $R$ be an associative ring with * such that $S$ has ACC on quadratic ideals. Then the nil radical of $S$ is nilpotent. If $S$ has DCC on quadratic ideals, then the nil radical of $S$ equals the prime radical.

**Proof.** We will first show that $N(S) = P(S)$ in either situation. We do this by showing that if $P$ is any proper prime ideal of $R$, then $P \supseteq N(S)$.

For, consider the ring $\overline{R} = R/P \cap P^*$ (possibly $P \cap P^* = P = P^*$); $\overline{R}$ is *-prime. Let $J = \overline{S}$, the image of $S$ in $\overline{R}$. Then $J$ is ample, has ACC or DCC on quadratic ideals, and so $N(J) = 0$ by the lemma. But $N(S) \subseteq N(J)$; thus, $N(S) \subseteq P \cap P^* \subseteq P$.

Now $P(R)$ is the intersection of all prime ideals of $R$, and so $N(S) \subseteq P(R)$. By [2, Theorem 3], $P(R) \cap S = P(S)$. But $N(S) \subseteq P(R) \cap S$ and $N(S) \supseteq P(S)$. Thus, $N(S) = P(S)$.

In the case when $S$ has ACC, $P(S)$ will simply be the maximal nilpotent ideal, so $N(S)$ is nilpotent.

**REFERENCES**


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