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CHAIN CONDITIONS ON SYMMETRIC ELEMENTS[1]

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ABSTRACT. Recently Britten has proven an analog of Goldie's theorem for the Jordan ring S of symmetric elements in a ring with involution of characteristic not 2. In this paper we first extend Britten's theorem to the situation where R is an arbitrary ring and the Jordan ring is only an ample subring of the symmetric elements. We apply this result to show that if S has ACC on quadratic ideals, then the (Jordan) nil radical of S is nilpotent.

Recently Britten [1] has proven an analog of Goldie's theorem for the Jordan ring of symmetric elements of a ring with involution. More specifically, he has shown that if $2R = R$ and the symmetric elements S of R are a prime Jordan ring with either ascending or descending chain condition on quadratic ideals, then $S \simeq S_1$, the symmetric elements of a *-prime ring R_1, and R_1 has a ring of quotients which is *-simple Artinian.

In this paper, we first extend Britten's theorem to the situation where R is an arbitrary ring and the Jordan ring is only an ample subring of the symmetric elements. The proof given here is considerably shorter than Britten's; the simplification comes from using an idea of Lanski [5] on direct sums of right ideals.

As an application, we show that if S satisfies the ascending chain condition on quadratic ideals, then the nil radical of S is nilpotent. This question remains open for general Jordan algebras.

Before beginning, we need some definitions. R will always denote an associative ring, and $*$ an involution on R (that is, an antiautomorphism of R of period 2). We let $S_R = \{x \in R | x^* = x\}$ denote the *symmetric* elements of R, $T_R = \{x + x^* | x \in R\}$ the *traces*, and $N_R = \{xx^* | x \in R\}$ the *norms*. An ideal I of R is called a **-ideal* if $I^* = I$; thus, R is **-prime* if the product of two nonzero *-ideals is nonzero, and **-simple* if there are no

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proper *-ideals. We say R is *semiprime* if R contains no nilpotent ideals; it is easy to see that a *-prime ring is semiprime.

In what follows, J will denote a special quadratic Jordan ring; that is, J will be an additive subgroup of R closed under the quadratic operator $xU_y = yxy$ and the binary composition x^2. A Jordan subring J of S_R is called *ample* if it contains all norms and traces and if $xJx^* \subseteq J$, for all $x \in R$. An additive subgroup $I \subset J$ is a *quadratic ideal* of J if $JU_I \subset I$.

We abbreviate the ascending (descending) chain condition by ACC (DCC). Recall that R is a right Goldie ring if R has ACC on right annihilators and R has no infinite direct sums of right ideals.

We begin with an easy lemma. Part (ii) is part of Britten's Lemma A [1].

Lemma 1. *Let R be a semiprime ring with *, and J an ample Jordan subring of S. Then:*

(i) *If $a \in J$ with $aJa = 0$, then $a = 0$.*

(ii) *If I is a right (left) ideal of R with $I \cap J = 0$, then $Q = \{x + x^* | x \in I\}$ is a quadratic ideal of J.*

Proof. (i) Since $a \in J$ and J is ample, $xax^* \in J$ for all $x \in R$. Thus, $axax^*a = 0$. On the other hand, $x + x^* \in J$ so $a(x + x^*)a = 0$. This gives $axa = -ax^*a$, and so $axaxa = 0$, all $x \in R$. But then $(ax)^3 = 0$; that is, aR is a nil right ideal of R of bounded index. By a theorem of Levitzki [3, p. 1], R has a nilpotent ideal (a contradiction), unless $a = 0$.

(ii) Say that I is a right ideal, and choose $a \in I$. Then $aJa^* \subseteq I \cap J = 0$. Thus, if $s \in J$, $a^*sa \in J$ and $a^*saJa^*sa = 0$. By (i), $a^*sa = 0$. Since s was arbitrary in J, $a^*Ja = 0$. It now follows that $JU_{a+a^*} \subseteq Q$, for all $a \in I$.

The first theorem extends Theorem 1 of Britten.

Theorem 1. *Let R be a *-prime ring and J ample. If J has no infinite direct sums of quadratic ideals, then R has no infinite direct sums of right or left ideals.*

Proof. Assume that the result is false, that is, J has no infinite direct sums of quadratic ideals but, in R, $\mathfrak{M} = \{T_i\}$ is an infinite collection of right ideals whose sum is direct.

We claim that we may assume (by replacing $\mathfrak{M}$ by an appropriate subset) that any finite sum of ideals in $\mathfrak{M}$ intersects J only in (0). For, let $\mathfrak{M}_n = \{T_n, T_{n+1}, \cdots\}$. Now, if $\mathfrak{M} = \mathfrak{M}_1$ fails, there exists a finite sum $S_1 = T_1 + T_2 + \cdots + T_{n_1 - 1}$ such that $S_1 \cap J \neq (0)$. Then try $\mathfrak{M}_{n_1} = \{T_{n_1}, \cdots\}$. If $\mathfrak{M}_{n_1}$ fails, there must be a finite sum $S_2 = T_{n_1} + \cdots + T_{n_2 - 1}$ with

$S_2 \cap J \neq (0)$. Repeat the argument: if $\mathfrak{M}_{n_k}$ fails, there is some $S_{k+1} \neq T_{n_k} + \cdots + T_{n_{k+1}-1}$ with $S_{k+1} \cap J \neq (0)$. But if S_k exists for all k, then the set $\{S_k \cap J\}$ is an infinite set of quadratic ideals of J which form a direct sum, a contradiction. Thus, for some k, it must be that in $\mathfrak{M}_{n_k}$, no finite sum of right ideals intersects J nontrivially. $\mathfrak{M}_{n_k}$ is the desired subset.

Now for each $T_i \in \mathfrak{M}$, consider $Q_i = \{x + x^* | x \in T_i\}$. By Lemma 1, Q_i is a quadratic ideal of J. Since J has no infinite direct sums of quadratic ideals, there is a finite set $\{Q_1, \cdots, Q_n\}$ which is dependent: for some $q_i \in Q_i$, not all zero, $\Sigma q_i = 0$. Write $q_i = x_i + x_i^*$ for $x_i \in T_i$ and let $a = \Sigma x_i \in \Sigma_{i=1}^n T_i$. Since there is no infinite direct sum in $\{Q_{n+1}, \cdots\}$, there must be another finite dependent set $\{Q_{n+1}, \cdots, Q_m\}$. Say that $\Sigma p_j = 0$ for not all $p_j \in Q_j$ zero. Let $p_j = y_j + y_j^*$ and $b = \Sigma y_j \in \Sigma_{n+1}^m T_j$. Then $a + a^* = b + b^* = 0$; thus a and b are skew and are nonzero by independence of the T_k.

Now let $r \in R$. Then $arb + br^*a \in (\Sigma T_i + \Sigma T_j) \cap J = (0)$, and so $arb = -br^*a$. But then $arb \in (\Sigma T_i) \cap (\Sigma T_j) = (0)$ by the directness of $\{T_k\}$. Thus, $arb = 0$, all $r \in R$. But then $(RaR)(RbR) = 0$, and RaR and RbR are *-ideals of R since a and b are skew. By the *-primeness of R, $a = 0$, or $b = 0$, a contradiction. Thus, R cannot have an infinite direct sum of right ideals.

Now if U is a subset of R, we write $\operatorname{ann}_r U = \{x \in R | Ux = 0\}$, the *right annihilator* of U. Similarly, $\operatorname{ann}_l U$ denotes the left annihilator of U. If $A_1 \subset A_2 \subset \cdots$ is a properly ascending chain of right annihilators in R, let $B_i = \operatorname{ann}_l A_i$. Then it is easy to verify that the B_i are a properly descending chain of left annihilators and that $A_i = \operatorname{ann}_r B_i$.

The next lemma is also part of Britten's Lemma A.

Lemma 2. *Let R be *-prime. Then if C is the right (left) annihilator of a nonzero ideal of R, $C \cap C^* = 0$.*

Proof. Let I be the ideal of R, so that $IC = 0$. Then CI is a right ideal of R with $(CI)^2 = 0$, and so $CI = 0$ since R is semiprime. Applying *, we have $I^*C^* = 0$ and thus $(I + I^*)(C \cap C^*) = 0$. Since $R(C \cap C^*)R$ is a *-ideal of R, we must have $C \cap C^* = 0$ since R is *-prime.

We are now able to extend Britten's Theorem 4. The argument is a simplification of his argument.

Theorem 2. *Let R be *-prime and J ample. If J has ACC or DCC on quadratic ideals, then R is a Goldie ring.*

Proof. If J has ACC or DCC on quadratic ideals, then it is easy to see

that J has no infinite direct sums of quadratic ideals. Thus, by Theorem 1, it suffices to show that R satisfies ACC on right annihilators.

We first treat the case when J has ACC on quadratic ideals. If R is not Goldie, let $A_1 \subset A_2 \subset \cdots \subset A_i \subset \cdots$ be a proper ascending chain of right annihilators in R, and let B_i be the corresponding descending chain of left annihilators.

First assume that $A_i \cap J \neq 0$ for some i. Since the A_i are ascending, we may as well assume that $A_i \cap J \neq 0$ for all i. Then $\{A_i \cap J\}$ is an ascending chain of quadratic ideals of J, so for some m, $A_m \cap J = A_{m+1} \cap J = \cdots$. We claim that this gives a contradiction. For, choose $a \in A_m \cap J$ and $y \in A_{m+1}$. Then $ya + ay^* \in A_{m+1} \cap J = A_m \cap J$, and so $0 = B_m(ya + ay^*) = B_m ya$, since $a \in A_m = \operatorname{ann}_r B_m$. Thus $(B_m A_{m+1})a = 0$. Since $B_m A_{m+1} \neq 0$, and $a \in J \subseteq S$, $a = 0$ by Lemma 2. Thus $A_m \cap J = 0$.

We may therefore assume that $A_i \cap J = 0$, for all i. For each A_i, let Q_i be given as in Lemma 1. Since the Q_i are ascending, for some m, $Q_m = Q_{m+1} = \cdots$. We claim that $A_{m+1} = A_m$. For, if not, choose $a \in A_{m+1}$, $a \notin A_m$. Then $a + a^* \in Q_{m+1} = Q_m$, so $a + a^* = a_m + a_m^*$, for some $a_m \in A_m$. Thus, $a - a_m = -(a - a_m)^* \in A_{m+i}$, $\notin A_m$, and is skew. So we may assume that a is skew. Now choose any $y \in A_{m+2}$. Then $ay^* + ya^* = ay^* - ya \in A_{m+2} \cap J = 0$, so $ay^* = ya$. But then $B_{m+1}ay^* = 0 = B_{m+1}ya$ since $a \in A_{m+1}$. That is, $(B_{m+1}A_{m+2})a = 0$. As above, this gives $a = 0$, a contradiction. Thus, $A_{m+1} = A_m$, which contradicts the A_i being a proper chain.

The proof for DCC is very similar; if $B_i \cap J = 0$ for all i, get the Q_i's as above. Then we can choose a skew in B_{m+1}, not in B_{m+2}. If $y \in B_m$, we again have $y^*a = ay$, so $a(B_m A_{m+1}) = 0$ and $a = 0$ as before. Thus, we may assume that $B_i \cap J \neq 0$, for some i. But the $\{B_i \cap J\}$ are a proper descending chain of quadratic ideals, so for some m, $B_m \cap J = B_{m+1} \cap J = \cdots$ and we may assume $B_m \cap J \neq 0$. But then, choosing $a \in B_m \cap J$ and $y \in B_m$, we see that $a(B_m A_{m+i}) = 0$, so $a = 0$.

We are now able to extend Britten's main theorem:

Corollary 1. *Let J be an ample subring of S_R, where R is a ring with $*$. If J is prime and has ACC or DCC on quadratic ideals, then J is isomorphic to an ample Jordan subring J_1 of a $*$-prime Goldie ring R_1.*

Proof. Let $P(R)$ denote the prime radical of R. Then by the same proof as [2, Lemma 1], $P(R) \cap J \subseteq P(J)$, the prime radical of J. But since J is prime, $P(J) = 0$; thus, $P(R) \cap J = 0$. Let $R_1 = R/P(R)$; R_1 is semiprime. Now J_1, the image of J in R_1, is isomorphic to J since $P(R) \cap J = 0$. Also J_1

is ample since J is (R_1 has an induced involution since $P(R)^* = P(R)$).

It remains only to show that R_1 is actually *-prime (since then R_1 Goldie follows from Theorem 2). Say $AB = 0$, where A, B are $\neq 0$ *-ideals of R. Then $A \cap J \neq 0$ (since if $A \cap J = 0$, $a + a^* = 0 = a^*a = a^2$, for all $a \in A$, and R would contain a nilpotent ideal). Similarly, $B \cap J \neq 0$. But $A \cap J$ and $B \cap J$ are ideals of J, and $(A \cap J)U_{B \cap J} = 0$, contradicting J being prime.

Since R_1 is a *-prime Goldie ring, by Goldie's theorem R_1 has a ring of quotients A which is semisimple Artinian. It is not difficult to see that in fact A is *-simple (see Britten's Theorem 5). We can therefore show that J has a Jordan ring of quotients.

Corollary 2. *Let J, R, J_1, and R_1 be as in Corollary 1. Then J has a Jordan ring of quotients $Q(J)$. If A is the associative ring of quotients for R_1, then $Q(J) \simeq Q(J_1)$ is an ample subring of S_A, and $Q(S_{R_1}) = S_A$. When $2R = R$, $Q(J)$ is a simple Jordan ring.*

Proof. Using Corollary 1, we may work entirely in R_1. We claim that every regular element of J is regular in R_1. For let $r \in J$, regular, and say that $rx = 0$ for some $x \in R_1$. Then $(x + x^*)U_r = 0$, so $x + x^* = 0$. Then $x = -x^*$, so $x^2 = -xx^* \in J$. Since $x^2U_r = 0$, $x^2 = 0$. But this says that the right annihilator of r is nil of index 2, and so R_1 contains a nilpotent ideal (a contradiction). Thus, $x = 0$. Similarly, $xr = 0$ implies $x = 0$.

We can now apply [7, Theorem 4.5], which states exactly that $Q(S_R) = S_A$, and $Q(J_1)$ is ample whenever R has a ring of quotients A and every element of J is regular in R.

The fact that when $2R = R$, $Q(J) \simeq S_A$ is simple follows from a theorem of Herstein [3, Theorem 2.6].

Having extended Britten's results to arbitrary characteristic, we now apply them to study the nil radical of S. Let $N(J)$ denote the nil radical of the Jordan ring J. As in Corollary 1, $P(R)$ will be the (associative) prime radical of R, and $P(J)$ the (Jordan) prime radical of J. We work with J, rather than S, since the symmetric elements might not be preserved under homomorphic images.

Lemma 3. *Let R be a *-prime ring in which J has either ACC or DCC on quadratic ideals. Then $N(J) = 0$.*

Proof. Since J is ample and $N(J)$ is an ideal of J, $N(J)$ is a core ideal

of S (for details see [6, p. 387]). Thus, if $N(J) \neq 0$, there exists a nonzero associative *-ideal B of R such that the core $K_0(B) \subseteq N(J)$, where $K_0(B) = \{b + b^* + \Sigma b_i b_i^* | b, b_i \in B\}$ [6, Corollary, p 387]. We claim that B must contain a nonzero nil right ideal. If $K_0(B) = 0$, this is trivial; for $b + b^* = 0$, or $b^* = -b$, for all $b \in B$. But then also $bb^* = -b^2 = 0$, and so B itself is nil. We may therefore assume that $K_0(B) \neq 0$. Choose $x \in K_0(B)$, $x \neq 0$. Then $x^n = 0$, $x^{n-1} \neq 0$ for some n, so letting $y = x^{n-1}$ we have $y^2 = 0$, $y \neq 0$. Now for any $b \in B$, $yb + b^*y \in K_0(B) \subseteq N(J)$, and so for some k, $(yb + b^*y)^k = 0$. Multiplying on the right by yb, we see $(yb)^{k+1} = 0$; that is, yB is nil.

By Theorem 2, R is a Goldie ring. By Lanski's theorem [4], any nil subring of R is nilpotent. Thus, B contains a nilpotent ideal. But B is an ideal in a semiprime ring, so is itself a semiprime ring, a contradiction. Thus, $N(J) = 0$.

Theorem 3. *Let R be an associative ring with * such that S has ACC on quadratic ideals. Then the nil radical of S is nilpotent. If S has DCC on quadratic ideals, then the nil radical of S equals the prime radical.*

Proof. We will first show that $N(S) = P(S)$ in either situation. We do this by showing that if P is any proper prime ideal of R, then $P \supseteq N(S)$. For, consider the ring $\overline{R} = R/P \cap P^*$ (possibly $P \cap P^* = P = P^*$); $\overline{R}$ is *-prime. Let $J = \overline{S}$, the image of S in $\overline{R}$. Then J is ample, has ACC or DCC on quadratic ideals, and so $N(J) = 0$ by the lemma. But $\overline{N(S)} \subseteq N(J)$; thus, $N(S) \subseteq P \cap P^* \subseteq P$.

Now $P(R)$ is the intersection of all prime ideals of R, and so $N(S) \subseteq P(R)$. By [2, Theorem 3], $P(R) \cap S = P(S)$. But $N(S) \subseteq P(R) \cap S$ and $N(S) \supseteq P(S)$. Thus, $N(S) = P(S)$.

In the case when S has ACC, $P(S)$ will simply be the maximal nilpotent ideal, so $N(S)$ is nilpotent.

REFERENCES

1. Daniel Britten, *On prime Jordan rings $H(R)$ with chain condition*, J. Algebra 27 (1973), 414–421.

2. T. S. Erickson and S. Montgomery, *The prime radical in special Jordan rings*, Trans. Amer. Math. Soc. 156 (1971), 155–164. MR 43 #306.

3. I. N. Herstein, *Topics in ring theory*, Univ. of Chicago Press, Chicago, Ill., 1969. MR 42 #6018.

4. C. Lanski, *Nil subrings of Goldie rings are nilpotent*, Canad. J. Math. 21 (1969), 904–907. MR 40 #1428.

5. C. Lanski, *Chain conditions in rings with involution* (to appear).

6. Kevin McCrimmon, *On Herstein's theorems relating Jordan and associative algebras*, J. Algebra **13** (1969), 382–392. MR **40** #2721.

7. Susan Montgomery, *Rings of quotients for a class of special Jordan rings*, J. Algebra **31** (1974), 154–165.

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