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MATRIX GROUP MONOTONICITY

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ABSTRACT. Matrices for which the group inverse exists and is nonnegative are studied. Such matrices are characterized in terms of a generalization of monotonicity. In particular, nonnegative matrices with this property are characterized in terms of their nonnegative rank factorizations.

1. **Introduction.** All matrices considered in this paper are real and square unless otherwise indicated.

A nonsingular matrix A and its inverse X satisfy the following matrix equations:

(1) $AXA = A$,

(2) $XAX = X$,

(3) $(AX)^T = AX$,

(4) $(XA)^T = XA$,

(5) $AX = XA$,

where "T" denotes the transpose. For a rectangular matrix A and for λ a subset of $\{1, 2, 3, 4\}$ containing 1, we say that X is a λ-inverse of A [1] if X satisfies equation i for each $i \in \lambda$. In particular the $\{1, 2, 3, 4\}$-inverse of A, the Moore-Penrose inverse, is unique and is denoted by A^+. A $\{1, 2\}$-inverse of A which satisfies (5) is necessarily square and is called [1] a group inverse. Matrices which have nonnegative λ-inverses are studied in [3], [5], [9]. In this paper we study matrices having a nonnegative group inverse. The group inverse of a matrix A does not always exist. When it exists it is unique and is denoted by $A^\#$(for example, $A = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ is a matrix for which $A^\#$ does not exist since A^2 is null). The existence of $A^\#$ is equivalent to the condition that rank A^2 = rank A which in turn is equivalent to the requirement that $\mathcal{R}(A) \cap \mathcal{N}(A) = \{0\}$, i.e., $\mathcal{R}(A)$ and $\mathcal{N}(A)$

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are complementary subspaces, where $\mathcal{R}(A)$ and $\mathcal{N}(A)$ denote the range and null space of A, respectively.

A rank factorization of an $m \times n$ matrix A of rank r is a factorization

$$A = BG \tag{6}$$

where B is $m \times r$ and G is $r \times n$. Another characterization of the existence of the group inverse of a square matrix A is given in [7] in terms of (6), namely, $A^{\#}$ exists if and only if GB is nonsingular. In this case

$$A^{\#} = B(GB)^{-2}G. \tag{7}$$

A matrix A is called EP, e.g. [11] (the etymology of this notation is unknown) if $\mathcal{R}(A) = \mathcal{R}(A^T)$. We remark that $A^{\#} = A^{+}$ if and only if A is EP.

In the next section, matrices which have a nonnegative group inverse are studied. The third and final section is devoted to nonnegative matrices with this property.

2. **Group monotonicity.** A square matrix A is *monotone* if it satisfies

$$Ax \geq 0 \Rightarrow x \geq 0. \tag{8}$$

Collatz has shown, e.g. [12], that A^{-1} exists and is nonnegative if and only if A is monotone. The concept was extended to rectangular $m \times n$ matrices in [3], where it was shown that $A^{+} \geq 0$, if and only if

$$Ax \in R_+^m + \mathcal{N}(A^T), \qquad x \in \mathcal{R}(A^T) \Rightarrow x \geq 0, \tag{9}$$

where R_+^m denotes the nonnegative orthant of the m-dimensional Euclidean space. A matrix A is called λ-*monotone* [5] if A has a nonnegative λ-inverse. Thus A satisfying (9) is $\{1, 2, 3, 4\}$-monotone. In addition, it was shown in [3] that A is $\{1, 4\}$-monotone if and only if it is row-monotone, that is

$$Ax \geq 0, \qquad x \in \mathcal{R}(A^T) \Rightarrow x \geq 0 \tag{10}$$

and A is $\{1, 3\}$-monotone if and only if A^T is row-monotone.

The importance of monotonicity and λ-monotonicity to results on convergence of iterative methods for linear systems is demonstrated in, among other places, [4] and [12].

We call a matrix A *group-monotone* provided that its group inverse exists and is nonnegative. In this section such matrices are characterized.

Theorem 1. *Let A be $n \times n$. Then A is group-monotone if and only if*

$$(11) \qquad Ax \in R^n_+ + \mathfrak{N}(A), \qquad x \in \mathfrak{R}(A) \Rightarrow x \geq 0.$$

Proof. Suppose $A^\# \geq 0$, $x \in \mathfrak{R}(A)$ and $Ax = u + v$, $u \geq 0$, $Av = 0$. Then $x = A^\# Ax = A^\# u + A^\# v = A^\# u \geq 0$.

Conversely assume that (11) holds. We show first that $A^\#$ exists. Suppose rank $A^2 <$ rank A. Then there exists a vector y such that $A^2 y = 0$ and

$$(12) \qquad x = Ay \neq 0.$$

Now $Ax = 0 \in R^n_+ + \mathfrak{N}(A)$, $x \in \mathfrak{R}(A)$ and so by (11) $x = Ay \geq 0$. Similarly $-x \geq 0$, contradicting (12).

To show that $A^\# \geq 0$, let $w \geq 0$ and decompose w into $w = u + v$, where $u \in \mathfrak{R}(A)$ and $Av = 0$. Such u and v can be chosen since $A^\#$ exists. Then since $u \geq -v$, $u \in R^n_+ + \mathfrak{N}(A)$. Thus $AA^\# u = u \in R^n_+ + \mathfrak{N}(A)$, $A^\# u \in \mathfrak{R}(A)$. By (11) then, $x = A^\# w = A^\# u \geq 0$ which completes the proof.

Notice that if A is EP then $A^\# = A^+$ and indeed (11) coincides with (9). In general, however, the nonnegativity of A^+ and $A^\#$ are not equivalent. As examples, let

$$A_1 = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \qquad A_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & 1 & 0 \end{pmatrix}.$$

Then

$$A_1^+ = \frac{1}{4}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 4 \\ 1 & 1 & 0 \end{pmatrix}, \qquad A_2^+ = \frac{1}{3}\begin{pmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

while

$$A_1^\# = \begin{pmatrix} 1 & -1 & 1 \\ 1 & -1 & 1 \\ -1 & 0 & -1 \end{pmatrix}, \qquad A_2^\# = A_2.$$

3. **Group inverses of nonnegative matrices.** It is well known that $A \geq 0$ and $A^{-1} \geq 0$ if and only if A is monomial, i.e., $A = DP$, where D is a diagonal matrix with positive diagonal entries and P is a permutation matrix.

Nonnegative λ-monotone matrices A are studied in [5]. In particular, nonnegative matrices A which have a nonnegative generalized inverse are studied in [3] and [9]. Nonnegative matrices which are self-inverse are characterized in [8], and those which are equal to their generalized inverse, in [2].

Let A be a nonnegative matrix. Then a nonnegative rank factorization of A is a factorization of the form (6), where B and G are nonnegative. Such a factorization does not always exist, e.g., [5], [6], and [10]. It is shown in [5] that if A is nonnegative and $\{1\}$-monotone (which is the case if it is nonnegative and group-monotone), then it has a nonnegative rank factorization and in every such factorization $A = BG$, G has a nonnegative right inverse and B has a nonnegative left inverse.

In this section we study nonnegative matrices which are group-monotone.

Theorem 2. *Let $A \geq 0$. Then A is group-monotone if and only if A has a nonnegative rank factorization $A = BG$, where GB is monomial. In this case every nonnegative rank factorization of A has this property.*

Proof. Suppose $A^{\#} \geq 0$. Then A is $\{1\}$-monotone and thus has a nonnegative rank factorization. Let $A = BG$ be any such factorization. Then GB is nonsingular and by (7), $A^{\#} = B(GB)^{-2}G$. Now by [5], B has a nonnegative left inverse B_L and G has a nonnegative right inverse G_R, and thus $(GB)^{-1} = GB(GB)^{-2} = GBB_L A^{\#} G_R \geq 0$, and since GB is nonnegative it is monomial.

Suppose $A = BG$ is a nonnegative rank factorization such that GB is monomial. Then $(GB)^{-2} \geq 0$ and by (7) $A^{\#}$ exists and is nonnegative, completing the proof.

We conclude with a theorem whose proof is similar to the proof of Theorem 2 and which reduces the study of nonnegative group monotone matrices to the study of nonsingular self-inverse nonnegative matrices, which were characterized in [8].

Theorem 3. *Let A be nonnegative. Then $A = A^{\#}$ if and only if A has a nonnegative rank factorization $A = BG$ such that $(GB)^{-1} = GB$. Moreover, if $A = A^{\#} \geq 0$ then every nonnegative rank factorization of A has this property.*

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