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CENTRAL LOCALIZATIONS OF REGULAR RINGS

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ABSTRACT. In this paper we show that a ring R is von Neumann regular (or a V-ring) if and only if every central localization of R at a maximal ideal of its center is von Neumann regular (or a V-ring). Strongly regular rings are characterized by the property that all central localizations at maximal ideals of the center are division rings. Also we consider whether regular PI-rings can be characterized by the property that all central localizations at maximal ideals of the center are simple.

Commutative von Neumann regular rings have been characterized in various ways. However, very few of these characterizations extend to noncommutative rings. The results in this paper arose from attempting to extend to noncommutative rings a well-known theorem of Kaplansky [11, Theorem 6] which states that commutative regular rings are characterized by the property that all localizations at maximal ideals are fields. It turns out that the obvious extension of this theorem to the noncommutative case is valid. That being, a ring is regular if and only if all central localizations at maximal ideals of the center are regular. An analogous theorem is obtained for V-rings. Also we show that strongly regular rings are characterized by the property that all central localizations at maximal ideals of the center are division rings.

With an eye to the commutative theory, we consider whether regular PI-rings can be characterized by the property that all central localizations at maximal ideals of the center are simple. We provide an example to show that this is not the case. However, it is true if and only if contraction provides a 1:1 correspondence between maximal ideals of the ring and maximal ideals of the center.

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Throughout this paper R will denote an associative ring which does have a unity and C will denote the center of R. If M is a prime ideal of C, then $C - M$ is a multiplicatively closed subset of C, and the ring of fractions of R with respect to $C - M$ can be defined in the usual way. This ring of fractions is denoted by R_M and is called the central localization of R at M. As a C-module, R_M is just the usual C-module localization of R. There is a canonical map $R \longrightarrow R_M$ given by $r \longrightarrow r/1$.

A ring R is called *von Neumann regular* (or just *regular*) if for each $a \in R$ there exists an $x \in R$ with $a = axa$. If R is a reduced ring, i.e., it has no nonzero nilpotent elements, and is also regular, then R is called *strongly regular*. R is called a *V-ring* if every simple right R-module is injective. By a *PI-ring* we mean a ring which satisfies a polynomial identity with coefficients in the centroid and at least one coefficient is invertible. If R is a PI-ring, then R_M satisfies the same PI as R [12, Lemma 1].

Lemma 1. *Let R be a ring with regular center C and let M be a maximal ideal of C. Then $MR = \mathrm{kernel}(R \longrightarrow R_M)$, $MR \cap C = M$, and $0 \longrightarrow MR \longrightarrow R \longrightarrow RM \longrightarrow 0$ is exact.*

Proof. First, $\mathrm{image}(R \longrightarrow R_M) = R_M$ because, if $c \in (C - M)$, then there exists $d \in C$ such that $(1 - dc)c = 0$ and $d \longrightarrow 1/c$. Let $A = \mathrm{kernel}(R \longrightarrow R_M)$. Since C is regular, C_M is a field [11, Theorem 6], and hence $A \cap C = \mathrm{kernel}(C \longrightarrow C_M)$ is a maximal ideal of C. Thus $A \cap C = M$ and so $MR \subseteq A$. If $a \in A$, then there exists $s \in (C - M)$ with $as = 0$. Because C is regular, we have $(1 - st)s = 0$ for some $t \in C$. Then $(1 - st) \in M$ and $a = (1 - st)a \in MR$. Therefore $A = MR$. This completes the proof.

A ring R is *fully* (*right*) *idempotent* if $I^2 = I$ for each (right) ideal I of R. Regular rings are fully right idempotent but not conversely [4, Theorem 1.4], [6, Examples 3.1 and 3.2].

Proposition 2. *A ring R is fully* (*right*) *idempotent if and only if R_M is fully* (*right*) *idempotent for each maximal ideal M of C.*

Proof. Let I be an (right) ideal of R. Since C_M is a flat C-module, $0 \longrightarrow (I^2)_M \longrightarrow I_M \longrightarrow (I/I^2)_M \longrightarrow 0$ is exact and I_M is an (right) ideal of R_M. Since R_M is fully (right) idempotent, $I_M = I_M^2 = (I^2)_M$. Thus $(I/I^2)_M = 0$ for each maximal ideal M of C. Therefore $I = I^2$ by [3, VII, Exercise 11] and R is fully (right) idempotent. Conversely, if R is fully (right) idempotent, then according to [10, Lemma 2.3] C is regular, and so $\mathrm{image}(R \longrightarrow R_M) = R_M$. Wherefore, R_M is fully (right) idempotent.

The next theorem is prompted by Kaplansky's conjecture that a semiprime ring is regular provided that all of its prime factor rings are regular [9, p. 2]. We obtain it as a corollary to Fisher and Snider's work on this conjecture [7].

Theorem 3. *The following statements are equivalent for a ring R with center C:*

(a) *R is regular.*

(b) *C is regular and R/MR is regular for each maximal ideal M of C.*

(c) *R_M is regular for each maximal ideal M of C.*

Proof. It is well known that the center of a regular ring is regular [15, Theorem 3], and hence (a) implies (b). Lemma 1 yields (b) implies (c). Suppose that each R_M is regular. Then R is fully idempotent by Proposition 2 and hence C is regular. By Fisher and Snider [7, Corollary 1.3], a fully idempotent ring is regular exactly when each of its prime factor rings is regular. So let P be a prime ideal of R. Then $P \cap C$ is a maximal ideal of C and $(P \cap C)R \subseteq P$. From Lemma 1, $R/(P \cap C)R \simeq R_{P \cap C}$ is regular. Wherefore R/P is regular and (a) results. This completes the proof of the theorem.

Remark. The ring R is called left *π-regular* if for each $a \in R$ there exists a positive integer n and an $x \in R$ with $a^n = xa^{n+1}$. The following facts imply that R is left π-regular if and only if R_M is left π-regular for each maximal ideal M of C.

(a) By [2, Lemma 1] C is π-regular provided R is left π-regular.

(b) As in Lemma 1, if C is π-regular, then $R \to R_M$ is onto and kernel$(R \to R_M) \subseteq MR$.

(c) By [7, Theorem 2.1] R is left π-regular provided all its prime factor rings are left π-regular.

Note that this adds another equivalence to Storrer [14, Lemma 5.6] and to Fisher-Snider [7, Theorem 2.3], that being, in the former, a commutative ring R is π-regular if and only if each R_M is π-regular, and, in the latter, a PI-ring R is π-regular exactly when each R_M is π-regular.

A ring is *biregular* if each of its principal ideals is generated by a central idempotent. (If R has no nonzero nilpotent elements, then it can be shown that R is biregular if and only if R_M is biregular for each maximal ideal M of C.)

Lemma 4. *The lattices of ideals of a biregular ring R, its center C, and its Boolean algebra of central idempotents B are isomorphic via the mappings*

$$A \to A \cap C \qquad (IR \leftarrow I),$$

$$A \to A \cap B \qquad (JR \leftarrow J).$$

Proof. Since an ideal A of R is determined by its central idempotents, the mappings $A \to A \cap C$ and $A \to A \cap B$ are injective. Let I be an ideal of C and let $c \in (IR \cap C)$. Then $c = \sum c_i x_i$, $c_i \in I$, $x_i \in R$. Since C is regular, $\sum Cc_i = Ce$ for some $e \in B \cap I$. Thus $c = ce \in I$ and $I = IR \cap C$. Whence $A \to A \cap C$ is surjective, and similarly $A \to A \cap B$ is surjective.

By extending the commutative property "all localizations at maximal ideals are fields" to "all central localizations at maximal ideals of the center are division rings," we obtain

Theorem 5. *A ring R is strongly regular if and only if R_M is a division ring for each maximal ideal M of C.*

Proof. We first note that the ring R is reduced exactly when each R_M is reduced. For if R is reduced and $(a/s)^2 = 0$ in some R_M, then $(a/1)^2 = 0$ and so $a^2t = 0$ for some $t \in C - M$. But then $(at)^2 = 0$ implies $at = 0$, and so $a/s = 0$. Thus R_M is reduced. Conversely, if each R_M is reduced and $a^2 = 0$, then $a/1 = 0$ in each R_M. Whence $a = 0$ and so R is reduced. Using Theorem 3 we now have that R is strongly regular if and only if each central localization R_M is strongly regular. Also, if R is strongly regular, then we have by Lemma 1 that $R_M \cong R/MR$ for each maximal ideal M of C. Because R is biregular we conclude, from Lemma 4, that R_M is a strongly regular simple ring, i.e., a division ring. This completes the proof.

By Kaplansky's theorem [11, Theorem 6] a commutative ring is a V-ring if and only if it is regular. In general, V-rings are not regular and vice-versa [4, Theorem 1.4], [6, Examples 3.1, 3.2 and 3.3]. It is true that V-rings are fully right idempotent [10, Corollary 2.2]. For an account of the relation between regular rings, V-rings, and fully right idempotent rings see Fisher [6].

Theorem 6. *A ring R is a V-ring if and only if R_M is a V-ring for each maximal ideal M of C.*

Proof. If R is a V-ring, then C is regular [10, Lemma 2.3] and R_M is a homomorphic image of R by Lemma 1. Therefore R_M is a V-ring. Conversely, if each R_M is a V-ring, then each R_M is fully right idempotent. Hence R is fully right idempotent by Proposition 2, and C is regular. But according to [6, Theorem 14] a fully right idempotent ring is a V-ring precisely when all its primitive factor rings are V-rings. Accordingly, let P be a primitive ideal of R. Then $P \cap C$ is a maximal ideal of C and $(P \cap C)R \subseteq P$. From Lemma 1, $R/(P \cap C)R \simeq R_{P \cap C}$ is a V-ring. Wherefore R/P is a V-ring.

Now we consider the question of whether regular PI-rings can be characterized as those for which all central localizations at maximal ideals of the center are simple. It follows immediately from Theorem 3 that a PI-ring R is regular if R_M is simple for each maximal ideal M of C. The following example of a regular PI-ring R with R_M not simple shows that the converse is false.

Example. Let R be the ring which consists of all sequences of 2×2 matrices with entries in a field F which are eventually diagonal. Then R is a regular PI-ring and C is isomorphic to $\Pi_{i=1}^{\infty} F$. Let M be a maximal ideal of C which contains $\bigoplus_{i=1}^{\infty} F$. We claim that R_M is not prime. Let $r_1 = \{\left[\begin{smallmatrix}1&0\\0&0\end{smallmatrix}\right]\}$ and $r_2 = \{\left[\begin{smallmatrix}0&0\\0&1\end{smallmatrix}\right]\}$. Then $r_1 1^{-1} \neq 0$ and $r_2 1^{-1} \neq 0$, but an easy computation shows that $r_1 1^{-1} R_M r_2 1^{-1} = 0$. Consequently R_M is not simple.

We can, however, characterize those regular PI-rings R for which each R_M is simple.

Theorem 7. *The following statements are equivalent for a PI-ring R with center C.*

1. *R_M is simple for each maximal ideal M of C.*
2. (a) *R is regular and*

(b) *the correspondence $M \to MR$ $(I \cap C \leftarrow I)$ is a $1:1$ correspondence between the maximal ideals of C and the maximal ideals of R.*

Proof. That (2) implies (1) follows immediately from Lemma 1. Suppose that (1) holds. Since R_M is a simple PI-ring [12, Lemma 1], it is regular by a theorem of Kaplansky. Consequently R is regular by Theorem 3. If M is a maximal ideal of C, then MR is a maximal ideal of R by Lemma 1. Thus, if I is a maximal ideal of R, then $I \cap C$ is a maximal ideal of C and so $I = (I \cap C)R$.

Corollary 8. *Let R be an Azumaya algebra which satisfies a PI. Then R is regular if and only if R_M is simple for each maximal ideal M of C.*

Proof. It follows from [5, Corollary 3.7, p. 54] that $M \to MR$ is a $1:1$ correspondence between maximal ideals of C and maximal ideals of R. Whence the result follows from Theorem 7.

Corollary 9. *Let R be a self-injective PI-ring. Then R is regular if and only if R_M is simple for each maximal ideal M of C.*

Proof. Armendariz and Steinberg [1, Theorem 3.5] show that a regular self-injective PI-ring is a finite direct sum of Azumaya algebras and hence is an Azumaya algebra. Thus an application of Corollary 8 and Theorem 7 completes the proof.

Corollary 10. *If R is a regular PI-ring for which* $\mathrm{Hom}_R(I, I)$ *is a projective R-module for each ideal I of R, then R_M is simple for each maximal ideal M of C.*

Proof. By Steinberg [13, Corollary 8] R is self-injective.

Corollary 11. *If R is a reduced PI-ring, then R is regular if and only if R_M is simple for each maximal ideal M of C.*

Proof. This follows from Theorems 5 and 7.

The Formanek center, $F(R)$, of a ring R is the ideal of C which consists of the values taken by all the Formanek central polynomials of R [8], [12, §3]. An ideal of either C or R is called identity-faithful [12, §3] if it does not contain $F(R)$. Rowen [12, Theorem 3] shows that for a semiprime PI-ring R, the correspondence $M \longrightarrow MR$ is a 1:1 correspondence between identity-faithful maximal ideals of R. Consequently, an application of Lemma 1 yields: *If R is a regular PI-ring, then R_M is simple for each identity-faithful maximal ideal M of C.*

Note that the Example shows that even if R is regular, $M \longrightarrow MR$ need not be a 1:1 correspondence between all maximal ideals of C and all maximal ideals of R, for this would contradict Theorem 7.

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