LINEAR SUPERPOSITION OF SMOOTH FUNCTIONS

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ABSTRACT. We give a simple proof of the impossibility of represent-
ing an arbitrary continuous function as a superposition (1), when
F_1, \ldots, F_N are smooth mappings of R^{n+1} to R^n. The main tool is the
Riemann-Lebesgue lemma.

Let \( V \) be an open subset of Euclidean space \( \mathbb{R}^{n+1} \), and \( F_1, \ldots, F_N \)
continuously differentiable mappings of \( V \) into \( \mathbb{R}^n \). Each \( N \)-tuple of bound-
ed continuous functions \( (g_1, \ldots, g_N) \) defined on \( \mathbb{R}^n \) determines a super-
position

\[
T(g_1, \ldots, g_N) = \sum_{k=1}^{N} g_k \circ F_k.
\]

This is an element of the Banach space \( C(V) \) of functions bounded and
continuous on \( V \).

Theorem. The range of the operator \( T \) is of first category in \( C(V) \),
whence the superpositions (1) do not exhaust \( C(V) \).

This theorem was proved for \( n = 2 \) by Vitushkin and Henkin [5]; in
fact they proved that the range of \( T \) is not even dense in \( C(V) \). For \( n = 3 \)
a stronger result was obtained by Fridman, but only for mappings of class
\( C^2(V) \) [2]. The theorem to be proved is neither implied by the work cited,
nor does it imply the strongest results known in special cases. See also
[1], [3], [4].

1. Let \( x_0 \) be a generic point for \( F_1, \ldots, F_N \), that is, the rank of each
Jacobian matrix \( J(F_k) \) attains a local maximum at \( x_0 \). Such points form
a dense \( G_\delta \) in \( V \). Then \( J(F_k) \) has rank \( d_k \) throughout a neighborhood \( W \)
of \( x_0 \) and (for an appropriate \( W \) \( F_k \) can be factored through \( R^{d_k} \):

Received by the editors October 15, 1973.

Key words and phrases. Smooth functions, Kolmogorov superposition theorem,
Baire category.

1The author is an Alfred P. Sloan Fellow.

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$G_k \circ H_k$, where $H_k$ maps $W$ into $R^{d_k}$. In the superpositions (1) we can replace $F_k$ by $H_k$, provided we allow functions $g_k$ defined on $R^{d_k}$. Let $L$ be a linear functional on $R^{n+1}$ whose Jacobian matrix contains a row not spanned by $J(F_k)(x_0)$ for each individual function $F_k$. On a small neighborhood, each mapping $H_k = (L, H_k)$ of $W$ into $R^{d_k+1}$ is absolutely continuous in the following sense: there is a nonnegative function $\phi_k$ on $R^{d_k+1}$ and an identity

$$\int_W f \circ H_k^*(x) \, dx = \int_{R^{d_k+1}} f(y) \phi_k(y) \, dy$$

whenever $f \geq 0$ is measurable on $R^{d_k+1}$. Using local coordinates on $R^{n+1}$ we can obtain a refinement of the absolute continuity: let $\psi$ be continuous and have compact support in $W$; then the function $\psi_k$ defined by $\int f \circ H_k^*(x) \psi(x) \, dx$ is continuous on $R^{d_k+1}$ (and has compact support).

2. For computations it is convenient to use coordinates $(t, u)$ in $R^{d_k+1}$: $t$ is real and $u$ is in $R^{d_k}$. We apply the change of variable formula above, with $f(t, u) = e^{i\lambda t} h(u)$; for the moment $h \in L^\infty(R^{d_k})$. This yields

$$\int e^{i\lambda L} h \circ H_k(x) \psi(x) \, dx = \int \int e^{i\lambda t} h(u) \psi_k(t, u) \, dt \, du.$$ 

We now suppose that $h = 0$ outside $H_k(W)$, since this leaves $H_k \circ h$ unchanged. The last integral has a modulus not exceeding

$$\|b\|_1 \sup \left| \int e^{i\lambda t} \psi_k(t, u) \, dt \right| = \|b\|_1 \cdot M_k(\lambda),$$

say. But $M_k(\lambda) \to 0$ as $\lambda \to +\infty$, because $\psi_k(t, u)$ has compact support in $R^{d_k+1}$—this is a simple extension of the Riemann-Lebesgue lemma. Because $\int e^{i\lambda L} \psi(x) \, dx = c > 0$ for all $\lambda$, we have a stronger version of the main theorem:

The set $T_1$ of superpositions $T(g_1, \ldots, g_N)$, with $\|g_k\|_1 \leq 1$ in $L^1(R^{d_k})$, has a closure $\overline{T}_1$ in $L^1(W)$, nowhere dense in $C(W)$.

REFERENCES


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