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LINEAR SUPERPOSITION OF SMOOTH FUNCTIONS

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ABSTRACT. We give a simple proof of the impossibility of representing an arbitrary continuous function as a superposition (1), when $F_1, \cdots, F_N$ are smooth mappings of R^{n+1} to R^n. The main tool is the Riemann-Lebesgue lemma.

Let V be an open subset of Euclidean space R^{n+1}, and $F_1, \cdots, F_N$ continuously differentiable mappings of V into R^n. Each N-tuple of bounded continuous functions $(g_1, \cdots, g_N)$ defined on R^n determines a *superposition*

$$T(g_1, \cdots, g_N) = \sum_1^N g_k \circ F_k. \tag{1}$$

This is an element of the Banach space $C(V)$ of functions bounded and continuous on V.

Theorem. *The range of the operator T is of first category in $C(V)$, whence the superpositions* (1) *do not exhaust $C(V)$.*

This theorem was proved for $n = 2$ by Vitushkin and Henkin [5]; in fact they proved that the range of T is not even dense in $C(V)$. For $n = 3$ a stronger result was obtained by Fridman, but only for mappings of class $C^2(V)$ [2]. The theorem to be proved is neither implied by the work cited, nor does it imply the strongest results known in special cases. See also [1], [3], [4].

1. Let x_0 be a generic point for $F_1, \cdots, F_N$, that is, the rank of each Jacobian matrix $J(F_k)$ attains a local maximum at x_0. Such points form a dense G_δ in V. Then $J(F_k)$ has rank d_k throughout a neighborhood W of x_0 and (for an appropriate W) F_k can be factored through R^{d_k}: $F_k =$

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$G_k \circ H_k$, where H_k maps W into R^{d_k}. In the superpositions (1) we can replace F_k by H_k, provided we allow functions g_k defined on R^{d_k}. Let L be a linear functional on R^{n+1} whose Jacobian matrix contains a row not spanned by $J(F_k)(x_0)$ for each individual function F_k. On a small neighborhood, each mapping $H_k^* = (L, H_k)$ of W into R^{d_k+1} is absolutely continuous in the following sense: there is a nonnegative function ϕ_k on R^{d_k+1} and an identity

$$\int_W f \circ H_k^*(x)\, dx \equiv \int_{R^{d_k+1}} f(y)\phi_k(y)\, dy$$

whenever $f \geq 0$ is measurable on R^{d_k+1}. Using local coordinates on R^{n+1} we can obtain a refinement of the absolute continuity: let ψ be continuous and have compact support in W; then the function ψ_k defined by $\int f \circ H_k^*(x)\, \psi(x)\, dx$ is continuous on R^{d_k+1} (and has compact support).

2. For computations it is convenient to use coordinates (t, u) in R^{d_k+1}: t is real and u is in R^{d_k}. We apply the change of variable formula above, with $f(t, u) = e^{i\lambda t}h(u)$; for the moment $h \in L^\infty(R^{d_k})$. This yields

$$\int e^{i\lambda L} h \circ H_k(x)\psi(x)\, dx = \iint e^{i\lambda t} h(u)\psi_k(t, u)\, dt\, du.$$

We now suppose that $h = 0$ outside $H_k(W)$, since this leaves $H_k \circ h$ unchanged. The last integral has a modulus not exceeding

$$\|h\|_1 \sup \left| \int e^{i\lambda t}\psi_k(t, u)\, dt \right| \equiv \|h\|_1 \cdot M_k(\lambda),$$

say. But $M_k(\lambda) \to 0$ as $\lambda \to +\infty$, because $\psi_k(t, u)$ has compact support in R^{d_k+1}—this is a simple extension of the Riemann-Lebesgue lemma. Because $\int |e^{i\lambda L}\psi(x)|\, dx = c > 0$ for all λ, we have a stronger version of the main theorem:

The set T_1 of superpositions $T(g_1, \cdots, g_N)$, with $\|g_k\|_1 \leq 1$ in $L^1(R^{d_k})$, has a closure $\overline{T}_1$ in $L^1(W)$, nowhere dense in $C(\overline{W})$.

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