SOME PROBLEMS ON $B_r$-COMPLETENESS

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ABSTRACT. In this paper, we give examples to show the following:
1. The product of two $B$-complete spaces is not necessarily $B_r$-complete.
2. The Mackey dual of a strict LF-space is not necessarily $B_r$-complete.
3. A separable, reflexive, and strict LF-space is not, in general, $B_r$-complete. The second point has reference to a problem of Dieudonné and Schwartz which asks essentially whether the Mackey dual of a strict LF-space is $B$-complete and which was answered in the negative by Grothendieck.

1. Introduction. Summers [8] and Iyahen [6] have shown that the product of two $B$-complete spaces is not necessarily $B$-complete. We give here two examples to show that the product of two $B$-complete spaces need not be $B_r$-complete, thus answering a question raised by Van Dulst [2], [3]. One of these is a space introduced by Grothendieck [4] and it gives us the first example of a separable, reflexive and strict LF-space (in fact the product of a separable and reflexive F-space and the direct sum of countable copies of a reflexive and separable Banach space) which is not $B_r$-complete. This is interesting since the countable direct sum of reflexive Banach spaces is $B$-complete. In [1], Dieudonné and Schwartz raised the following problem:

Let $E = \lim \text{ind } E_n$ be a strict LF-space. Let $H$ be a subspace of $E$ such that $H \cap E_n$ is closed for all $n$. Is $H$ necessarily closed?

In the light of Corollary 2 below, this is equivalent to asking whether the Mackey dual of a strict LF-space is $B$-complete. Grothendieck [4] gave a counterexample, but it did not answer the following more inclusive question:

Is the Mackey dual of a strict LF-space even $B_r$-complete?

The same example as mentioned above is used here to show that the answer to the above question is negative in general.

The other example (a space introduced by Köthe [7]) further shows that
2. Preliminary results. In this section, we give some criteria for $B^*$- and $B^*_r$-completeness which will be used in the sequel. Notations and terminology may be found in [5].

**Theorem 1.** Let $(E, u)$ be a quasi-complete locally convex space such that each absolutely convex compact subset is metrizable. Then $(E', r)$ (where $r$ is any topology lying between the topology $c$ of uniform convergence on absolutely convex compact subsets of $(E, u)$ and the Mackey topology $\tau(E', E)$) is $B$-complete ($B^*_r$-complete) iff every sequentially closed (dense) subspace of $(E, u)$ is closed.

**Proof.** It is sufficient to prove the result for $(E', c)$. Now $(E', c)' = E$. The condition of metrizability implies that each sequentially closed subspace of $(E, u)$ is almost closed.

Conversely, let $L \subseteq E$ be almost closed and $\{x_n; n \in \mathbb{N}\} = K$ be a sequence in $L$ converging to $x \in E$. Then the closed absolutely convex hull $B$ of $K$ is compact by quasi-completeness and $L \cap B = L \cap (B^0)^0$ is closed since, by hypothesis, $L$ is almost closed. But this implies that $x \in L$ and hence $L$ is sequentially closed.

**Note.** In the above, it is clear that quasi-completeness can be replaced by the following condition:

The closed absolutely convex hull of every convergent sequence is compact.

It should be noted that sequentially closed subspaces need not, in general, be almost closed. Let $E = \prod_{\alpha \in \Lambda} l_2^\alpha$, where each $l_2^\alpha$ is a copy of $l_2$ and cardinality $|\Lambda| \geq c$ (where $c$ is the cardinality of the continuum). It is easily seen that if $H$ is the subspace of $E$ consisting of all elements of $E$ which vanish for all but countable values of $\alpha$, then $H$ is sequentially closed but not almost closed.

**Corollary 1.** Let $(E, u)$ be a quasi-complete locally convex space such that $(E', c)$ is separable. Then $(E', r)$, $c \leq r \leq \tau(E', E)$, is $B$-complete ($B^*_r$-complete) iff every sequentially closed (dense) subspace of $(E, u)$ is closed.

**Proof.** The condition implies that the absolutely convex compact subsets are weakly metrizable and hence metrizable in the original topology $u$. 

Corollary 2. Let \((E, \tau)\) be any strict LF-space. Then \((E', \tau)\) (same notation as above) is \(B\)-complete (\(B_r\)-complete) iff every sequentially closed (dense) subspace of \((E, \tau)\) is closed.

Corollary 3 (Ilyachen [6]). Let \(E = A \times B\) where \(A\) is the direct sum of countable copies of \(l_p\) and \(B\) is the product of countable copies of \(l_q\) with \(q < p\) and \(1/p + 1/q = 1\). Then \(E\) is not \(B\)-complete.

Proof. Grothendieck [4], in fact, shows that \(E\) contains a sequentially closed subspace \(H\) which is not closed. Since \(E\) is an LF-space, it follows by Corollary 2 that \((E', \tau(E', E))\) is not \(B\)-complete. But \((E', \tau(E', E))\) is topologically isomorphic to \((E, \tau)\).

Note. It is possible to state results similar to Theorem 1 for locally convex spaces satisfying the Krein-Schmulian theorem or for hypercomplete locally convex spaces by substituting the words “convex set” or “absolutely convex set”, respectively, in place of the word “subspace.”

3. Counterexamples. The first counterexample, given below in Theorem 2, uses a space introduced in [4].

Theorem 2. Let \(A = \bigoplus_{n=1}^{\infty} l_p^n\) and \(B = \prod_{n=1}^{\infty} l_q^n\), where \(l_p^n\) and \(l_q^n\) are copies of \(l_p\) and \(l_q\) respectively with \(1 < q < p\) and \(1/p + 1/q = 1\). Let \(G = A \times B\).

Then
(a) \(G\) is a strict LF-space.
(b) \(A\) and \(B\) satisfy the Krein-Schmulian theorem and, hence, are \(B\)-complete.
(c) \(G\) is not \(B_r\)-complete.
(d) \((G', \tau(G', G))\) is not \(B_r\)-complete.
(e) \(G\) contains a sequentially closed dense subspace \(H\) which, even with the Mackey topology \(\tau(H, H')\), is not bornological.

Proof. (a) is clear. In fact, \(G = \lim\) ind \(F_n\), where \(F_n = (l_p \times l_p \times \cdots \times l_p) \times B\) (\(n\) factors of \(l_p\)).

(b) From Corollary 2 and the remarks following, and since \(A\) is the Mackey dual of the Fréchet space \(B\), it follows that \(A\) satisfies the Krein-Schmulian theorem. \(B\) also satisfies it since \(B\) is a Fréchet space.

(c) As in Corollary 3, it is sufficient to prove (d).

(d) By Corollary 2, it suffices to construct a sequentially closed dense subspace \(L\) in \(G\) which is not closed.

Construction. Let \(e_k = \{\delta_{kn}n\}_{n \in N}\) where...
\[
\delta_{kn} = \begin{cases} 
0 & \text{if } k \neq n, \\
1 & \text{if } k = n.
\end{cases}
\]

Then \(e_k \in l_p \cap l_q\). Define \(e_{ik} = \{\delta_{in} e_k\}_{n \in N}\). Then \(e_{ik}\) lies in \(A \cap B\). Let 
\[a_{ik} = (e_{ik}, e_{i-1,k}).\]
Define 
\[
b_{jkn} = (x_{jkn}, e_j + ((jkn)!)((jkn)!)e_{2j,\psi(k,n)}),
\]
where \(\psi: N \times N \to N\) is one-one and onto. We next define \(x_{jkn}\). Let \(\theta: N \times N \times N \to N\) be one-one and onto. Let \(S_{kn} = \theta((k, n) \times N) \subseteq N\). Then each \(S_{kn}\) is a countably infinite set and \(S_{kn} \cap S_{rt} = \emptyset\), if \((k, n) \neq (r, t)\).

Choose an element \(x_{kn} \in l_p - l_q\) (set-theoretic difference) for each \((k, n) \in N \times N\) such that \(x_{kn}\) is zero outside \(S_{kn}\) and has \(p\)-norm not exceeding \(1/n\). Such a choice is possible since \(l_q\) is a dense proper subspace of \(l_p\).

Define 
\[
x_{jkn} = \{\delta_{jr} x_{kn}\}_{r \in N}.
\]

Then \(x_{jkn} \in A\) and \(\lim_{n \to \infty} \|x_{jkn}\|_p = 0\) where \(\|\cdot\|_p\) denotes the usual \(p\)-norm.

Let \(L\) be the set of all elements \(x\) of \(G\) which are of the form:

\[
x = \sum_{i,k=1}^{\infty} \mu_{ik} a_{ik} + \sum_{j,k,n=1}^{\infty} \lambda_{jkn} b_{jkn},
\]

where \(\mu_{ik}\) and \(\lambda_{jkn}\) are scalars and

\[
\sum_{k=1}^{\infty} |\mu_{ik}|^q < \infty \quad \text{for each } i,
\]

\[
\sum_{k,n=1}^{\infty} |\lambda_{jkn}|^q ((jkn)!)((jkn)!)^{-q} < \infty \quad \text{for each } j,
\]

and there exists for each such \(x\) a fixed \(i_0\) (depending on \(x\)) such that

\[
\mu_{ik} = \lambda_{jkn} = 0 \quad \text{for } i, j > i_0.
\]

From (3), it follows easily that

\[
\sum_{n} |\lambda_{jkn}| < \infty \quad \text{for each fixed } j \text{ and } k,
\]

and

\[
\sum_{n} |\lambda_{jkn}| < \infty \quad \text{for each fixed } j \text{ and } k,
\]
Let $i_0$ and $j_0$ be the least values of $i$ and $j$, respectively, in (1) such that $\mu_{ik} = 0$ for $i > i_0$, and $\lambda_{jkn} = 0$ for $j > j_0$. Let $z_0 = \max(i_0, j_0)$.

Following Köthe [7], we may write the elements $x$ of $G$ in the form:

\[(6) \quad x = (\cdots, x_{-n}, \cdots, x_{-2}, x_{-1}||x_1, \cdots, x_n, \cdots)\]

where $x_i \in l_q$ and $x_{-i} \in l_p$ and all but a finite number of $x_{-i} = 0$.

If $x$ in (1) is written in the above form, then it is not difficult to see that $x_{-z_0} \neq 0$.

This is clear, if $i_0 \neq j_0$. In case $i_0 = j_0$, the same result is true since the sum of an element in $l_p - l_q$ and an element in $l_q$ cannot be zero.

Let

\[(7) \quad x_r = \sum_{i \leq m-1, k} \mu_{ik}^r a_{ik} + \sum_{j \leq m-1, k, n} \lambda_{jkn}^r b_{jkn} \in L\]

converge to an element $x \in G$. If $i_r$ and $j_r$ are defined for $x_r$ (as $i_0$ and $j_0$ for $x$ above), then convergence of $x_r$ implies, by the properties of the inductive limit topology of LF-spaces [1, Proposition 4], that there exists a fixed positive integer $j_0$ such that

\[z_r = \max(i_r, j_r) < j_0 \quad \text{for all } r.\]

Let $L_m$ be the set of all such sums in (1) for which $\mu_{ik} = \lambda_{jkn} = 0$ for $i, j > m$ and any $k, n$. Then we have $L = \bigcup \{L_m; m \in \mathbb{N}\}$. In order to show that $L$ is sequentially closed, it suffices to show that $L_m$ is sequentially closed for all $m \geq 1$. This we do by induction. Assume $L_{m-1}$ sequentially closed and let $x_r$ as in (7) be a sequence in $L_m$ converging to $x \in G$. Then we can write $x_r$ in the following form:

\[x_r = \sum_{i \leq m-1, k} \mu_{ik}^r a_{ik} + \sum_{j \leq m-1, k, n} \lambda_{jkn}^r b_{jkn} + x_r' \quad \text{(say)}\]

where

\[x_r' = \sum_k \mu_{mk}^r a_{mk} + \sum_{k, n} \lambda_{mkn}^r b_{mkn}.\]

Let $x_r$ be represented in the form (6) and let $x_t^r$ denote the element in the $t$th position where $t$ is a positive or negative integer. The convergence of the sequence $x_r$ implies the convergence in $l_q$ of the sequence $x_{2m}^r$. 

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Since
\[ x_{2m}^r = \sum_{k,n} \lambda_{mkn}^r d_{mkn} e_{2m, \psi(k,n)} \]
(where \( d_{mkn} \) equals \( (mkn)! \) raised to the power \( (mkn)! \)) converges in \( l_q \)
and since convergence in \( l_q \) implies coordinate-wise convergence, it follows that the limit element \( \bar{x}_{2m} \) is of the form:
\[ \bar{x}_{2m} = \sum_{k,n} \lambda_{mkn} d_{mkn} e_{2m, \psi(k,n)} \]
where \( \lambda_{mkn} = \lim_{r} \lambda_{mkn}^r \) for each \( k, n \). This implies that for any \( \epsilon > 0 \), there exists a positive integer \( R \) such that for \( r > R \), we have
\[ |x_{2m}^r - \bar{x}_{2m}|_q = \sum_{k,n} |\lambda_{mkn}^r - \lambda_{mkn}|_q |d_{mkn}|_q < \epsilon. \]

Similarly, for the sequence \( x'_r \), the element \( x''_{mr} \) at the \( m \)th position is given by:
\[ x''_{mr} = \sum_{k,n} \lambda_{mkn} e_{mk} = \sum_{k} \left( \sum_{n} \lambda_{mkn}^r \right) e_{mk}. \]
(Recall by (4) \( \sum_{n} |\lambda_{mkn}^r| \) is convergent.) Using inequality (8), it can be shown without much difficulty that the sequence \( x''_{mr} \) converges to \( x'_m \) in \( l_q \) where \( \bar{x}'_m = \sum_{k} (\sum_{n} \lambda_{mkn}^r) e_{mk} \).

Again, the convergence of the sequence \( x'_r \) in \( G \) implies that the sequence \( x''_{2m-1} \) converges in \( l_q \). Since \( x''_{2m-1} = \sum_k \mu_{mk} e_{2m-1,k} \), it follows by an argument essentially similar to that for the sequence \( x''_{2m} \), that the limit element \( \bar{x}_{2m-1} \) is of the form \( \sum_k \mu_{mk} e_{2m-1,k} \), where \( \mu_{mk} = \lim_{r} \mu_{mk}^r \).

Let us next look at the \(-m\)th position. Using the same notation,
\[ x''_{-m} = \sum_k \mu_{mk} e_{mk} + \sum_{k,n} \lambda_{mkn}^r x''_{mn}. \]
The sequence \( \sum_k \mu_{mk}^r e_{mk} \) certainly converges in \( l_p \) since we have seen that it converges in \( l_q \), and the limit element is \( \sum_k \mu_{mk} e_{mk} \). Also
\[ \left| \sum_{k,n} \lambda_{mkn}^r x_{mkn} - \sum_{k,n} \lambda_{mkn} x_{mkn} \right|_p = \sum_{k,n} |\lambda_{mkn}^r - \lambda_{mkn}|_p |x_{mkn}|_p \]
\[ \leq \sum_{k,n} |\lambda_{mkn}^r - \lambda_{mkn}|_p, \]
but the right-hand side tends to zero by the inequality (8).

The above arguments thus imply that the sequence \( x'_r \) converges in \( G \).
to the element $\sum_k \mu_{mk} a_{mk} + \sum_{k,n} \lambda_{mkn} b_{mkn} \in L_m$ (since it can be seen without much difficulty that conditions (2) and (3) are satisfied). Hence, by the induction hypothesis, it follows that the sequence $x_r - x_r'$ in $L_{m-1}$ converges to some element in $L_{m-1}$ and hence, the sequence $x_r$ itself converges to some element in $L_m$.

A similar type of argument shows that $L_1$ is also sequentially closed. Note that since the relative topology in the finite products of $l_p$ and $l_q$ which contains $L_m$ is metric, it is clear that $L_m$ is closed in $G$ for all $m$.

That $L$ is proper is clear since in $G$ there exist elements of the form $(\cdots, x_{-n}, \cdots, x_{-2}, x_{-2}, x_{-1}, x_1, x_2, \cdots, x_n, \cdots)$ where $x_n \neq 0$ for infinitely many values of $n$.

We next show that $L$ is dense. Let $E_j$ be defined as follows: $E_j = A \times l_q \times \cdots \times l_q$ ($E_j$ has $j$ factors of $l_q$). Let $\pi_j$ be the projection map of $G$ onto $E_j$. Then inductively, it can be seen as in [7] (by using the fact that $\lim_{n \to \infty} x_{jkn} = 0$ in $l_p$ for each $j, k$) that $(\pi_j L)^0 = 0$ for each $j$. This clearly implies that $L$ is dense.

(e) Let $H = L \oplus [x_1]$ where $x_1$ is an element of $G$ such that $x_1 \notin L$ and $[x_1]$ is the subspace generated by $x_1$. It is easy to show that $H$ is also sequentially closed in $G$. Let $f \neq 0$ be a linear functional on $H$ with the kernel $L$. Then by Webb [9], $f$ is sequentially continuous. If $H$ with its Mackey topology is bornological, then $f$ is continuous on $H$ and so has a unique continuous extension $\overline{f}$ to $G$. But then the kernel $N(\overline{f})$ of $\overline{f}$ is a closed subspace of $G$ containing $L$ and as $L$ is dense, it follows that $N(\overline{f}) = G$. This means that $\overline{f} = 0$, which is a contradiction. This completes the proof of Theorem 2.

The other counterexample is given below in Theorem 3.

**Theorem 3.** Consider the space $\phi \omega \times \omega \phi$ given in [7]. Then we have the following:

(a) $\phi \omega$ and $\omega \phi$ are separable, nuclear and $B$-complete locally convex spaces.

(b) $\phi \omega \times \omega \phi$ is a separable, nuclear and complete locally convex space.

(c) $\phi \omega \times \omega \phi$ is not $B_r$-complete.

**Proof.** Clearly, $\phi \omega$ and $\omega \phi$ are separable, nuclear and complete locally convex spaces. Now the Mackey dual of $\phi \omega$ is $\omega \phi$ and that of $\omega \phi$ is $\phi \omega$. Köthe [7] shows that any sequentially closed subspace in these spaces is always closed. So, by Corollary 1, $\phi \omega$ and $\omega \phi$ are $B$-complete.

(b) This is clear from (a).
(c) Köthe [7] has constructed a dense and sequentially closed subspace in the space $\phi_\omega \times \omega \phi$ which is not closed. If $E$ denotes the space $\phi_\omega \times \omega \phi$, then by (a), $(E', r(E', E))$ is topologically isomorphic to $E$, and since $E$ is separable, it follows by Corollary 1 that $(E', r(E', E))$ and hence $E$ is not $B_r$-complete.

I would like to express my thanks to Professor A. C. Cochran and Professor W. H. Summers for many helpful discussions. The latter drew my attention to Köthe's work in [7].

REFERENCES


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