ABSTRACT. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space, \(\mu(\Omega) < \infty\). Let \(X_n\) be a sequence of measurable functions on \(\Omega\) taking values in a compact metric space \(M\). The set of bounded stopping times \(\tau\) for the \(X_n\) is a directed set under the obvious ordering. The following theorem is proved: \(X_n\) converges pointwise almost everywhere if and only if the generalized sequence \(\int \phi(X_n) \, d\mu\) converges for every continuous function \(\phi\) on \(M\). The martingale theorem is proved as an application.

1. The general result. Let \((\Omega, \mathcal{F}, \mu)\) be a measure space, \(\mu(\Omega) < \infty\). Let \(M\) be a compact metric space, \(C(M)\) the space of continuous functions on \(M\). Let \(X_n = \Omega \rightarrow M\) be measurable, \(n = 1, 2, \cdots\). Define \(\mathcal{F}_n = \mathcal{F}(X_1, \cdots, X_n)\). A map \(\tau: \Omega \rightarrow \{1, 2, \cdots\}\) will be called a stopping time if \(\{\tau = n\} \in \mathcal{F}_n\) for all \(n\). Let \(\Gamma\) be the collection of bounded stopping times \(\tau\).

Definition 1. A map \(f: \Gamma \rightarrow \mathbb{R}\) will be called a generalized sequence. We write \(f(\tau) = a_\tau\). A generalized sequence \(a_\tau\) will be said to converge to a number \(a\) if for every \(\varepsilon > 0\) there exists \(\sigma \in \Gamma\) such that \(|a_\tau - a| < \varepsilon\) for \(\tau \geq \sigma\) (cf. [2, I.7.1]). Clearly we may choose \(\sigma = n\) everywhere for some \(n\). It is easy to see that a generalized sequence \(a_\tau\) converges if and only if for every strictly increasing sequence \(\tau(n) \in \Gamma\) the ordinary sequence \(a_{\tau(n)}\) converges.

The basic result is Theorem 1. A similar theorem in a continuous-time setting appears in Meyer [3, Proposition 6(a), p. 232]. References to results of the same sort by F. Mertens and M. Rao are also given in [3].

Theorem 1. The following two statements are equivalent:

(i) \(X_n\) converges pointwise almost everywhere on \(\Omega\).

(ii) For every \(\phi \in C(M)\), \(\int \phi(X_n) \, d\mu\) is a convergent generalized sequence.

Proof. (i) \(\Rightarrow\) (ii). Fix \(\phi \in C(M)\). If (i) holds then \(\phi(X_n)\) converges pointwise almost everywhere on \(\Omega\). The usual proof of Lebesgue’s bounded convergence theorem applies without any real change.
(ii) $\Rightarrow$ (i). Let us first consider the case $M = [0, 1]$. Let (ii) hold and (i) not hold. We will obtain a contradiction. Let $X^- = \limsup X^n$, $X_+ = \liminf X^n$. Then $\mu(X_+ < X^-) > 0$, so there exist $\alpha$ and $\beta$ such that $\mu(X_+ < \alpha < \beta < X^-) > 0$. Choose $\epsilon > 0$ such that $\mu(X_+ < \alpha < \beta < X^-) > \epsilon$. Define $\phi \in \mathcal{C}(M)$ so that $\phi = 0$ on $[0, \alpha]$, and $\phi = 1$ on $[\beta, 1]$. It is easy to see that for any $n$ we can find bounded stopping times $\sigma$, $\gamma$ such that $\gamma \geq \sigma \geq n$, $\{X_\sigma \geq \alpha \} \subseteq \{\sigma = \gamma\}$, and $\mu(X_\sigma < \alpha < \beta < X_\gamma) > \epsilon$. Then $\int \phi(X_\gamma) > \int \phi(X_\sigma) + \epsilon$. Hence the sequence $\int \phi(X_n)$ cannot converge, and a contradiction is obtained. Thus (ii) $\Rightarrow$ (i) is proved, for the case $M = [0, 1]$. This is the standard case, since $[a, b]$ is homeomorphic to $[0, 1]$ for $-\infty < a < b < \infty$.

In the general case, if (ii) holds we use the preceding result to show that $\phi(X_n)$ converges pointwise almost everywhere on $\Omega$, for each $\phi$ in $\mathcal{C}(M)$. Let $\{\phi_i\}$ be a countable dense subset of $\mathcal{C}(M)$. There exists $\Omega_1 \subseteq \Omega$ such that $\mu(\Omega - \Omega_1) = 0$ and $\phi_i(X_n(\omega))$ converges as $n \to \infty$ for each $\omega \in \Omega_1$, and all $i$. Hence $\phi(X_n(\omega))$ converges as $n \to \infty$ for each $\omega \in \Omega_1$, and all $\phi$ in $\mathcal{C}(M)$. It follows that $X_n(\omega)$ converges for each $\omega$ in $\Omega_1$. This completes the proof of Theorem 1.

Definition 2. A collection $H \subseteq \mathcal{C}(M)$ will be called a separating class if for any bounded Borel measures $\mu$ and $\nu$ on $M$, $\int \phi \, d\mu = \int \phi \, d\nu$ for all $\phi \in H$ implies $\mu = \nu$. A collection $K$ of Borel functions on $M$ will be said to generate a separating class if the set of all linear combinations of functions in $K$ contains a separating class in $\mathcal{C}(M)$. The functions in $K$ need not be continuous.

Corollary to Theorem 1. The following condition is equivalent to (i) and (ii) of Theorem 1:

(iii) $\int \phi(X_n(\omega)) \, d\mu$ exists and is convergent for each $\phi$ in a collection $K$ that generates a separating class in $\mathcal{C}(M)$.

Proof. It is enough to show (iii) $\Rightarrow$ (ii). We may assume $K$ is a separating class in $\mathcal{C}(M)$. The proof is then the same as that for ordinary sequences, applied to $X_{\tau(n)}$ for each strictly increasing sequence $\tau(n) \in \Gamma$.

Example 1. Let $M = [-\infty, \infty]$. Let $K_0$ be the collection of functions $\phi$, continuous on $(-\infty, \infty)$, such that:

1. $\phi$ is constant on $(-\infty, a]$ and $\phi$ is linear and increasing on $[a, \infty)$, for some $a$, and
2. $\phi$ is arbitrary at $\pm \infty$.

Then $K_0$ generates a separating class in $\mathcal{C}(M)$, since the characteristic function of any interval $[b, \infty)$ is a bounded pointwise limit of differences of members of $K_0$. 

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It is clearly desirable to have to check the convergence of $\int \phi(X_r) \, d\mu$ for as few $\phi$'s as possible. The following observation is useful in this respect, though it will not be needed in §2.

**Lemma.** Let $\phi$ and $\theta$ be Borel functions on $M$ such that $\int \phi(X_r) \, d\mu$ and $\int \theta(X_r) \, d\mu$ converge and such that $\int \phi \vee \theta(X_r) \, d\mu$ is bounded above. Then $\int \phi \vee \theta(X_r) \, d\mu$ converges.

**Proof.** Given $\epsilon > 0$, choose $N$ such that $r, \sigma \geq N$ implies $|\int \phi(X_r) - \int \phi(X_\sigma)| < \epsilon$ and $|\int \theta(X_r) - \int \theta(X_\sigma)| < \epsilon$. Choose $r \geq N$ such that $\int \phi \vee \theta(X_\sigma) < \int \phi \vee \theta(X_r) + \epsilon$ for any $\sigma \geq N$. Now suppose $\sigma \geq r$. Choose Borel sets $A$ and $B$ such that $A \cap B = \emptyset$, $A \cup B = M$, $\phi \vee \theta = \phi$ on $A$, and $\phi \vee \theta = \theta$ on $B$. Define the stopping time $\sigma_1$ by $\sigma_1 = \sigma$ on $X_r^{-1}(A)$ and $\sigma_1 = r$ on $X_r^{-1}(B)$. Clearly $\int \phi(X_r) < \int \phi(X_{\sigma_1}) + \epsilon$. Then

$$\int_{X_r^{-1}(A)} \phi \vee \theta(X_r) < \int_{X_r^{-1}(A)} \phi \vee \theta(X_{\sigma_1}) + \epsilon.$$  

Similarly

$$\int_{X_r^{-1}(B)} \phi \vee \theta(X_r) < \int_{X_r^{-1}(B)} \phi \vee \theta(X_{\sigma_1}) + \epsilon.$$  

Thus $\int \phi \vee \theta(X_r) < \int \phi \vee \theta(X_{\sigma_1}) + 2\epsilon$, so that

$$|\int \phi \vee \theta(X_r) - \int \phi \vee \theta(X_{\sigma_1})| < 2\epsilon,$$  

and the Lemma is proved.

2. The martingale theorem. Let $M = [-\infty, \infty]$. The sequence $X_n$ of §1 will be called a submartingale if

(a) $E(|X_n|) < \infty$ for all $n$, and

(b) $E(X_{n+1}|X_n, \cdots, X_1) \geq X_n$ for all $n$.

We wish to prove the martingale theorem [1, Chapter 7]:

**Theorem 2.** Let $E(|X_n|)$ be bounded. Then $X_n$ converges pointwise almost everywhere.

**Proof.** (a) and (b) imply by the submartingale stopping theorem that for $r, \sigma \in \Gamma$ with $r \geq \sigma$ we have $E(X_r|X_\sigma) \geq X_\sigma$. Then by Jensen's inequality we have $E(\phi(X_r)) \geq E(\phi(X_\sigma))$ for any $\phi \in \mathcal{K}_0$, where $\mathcal{K}_0$ is defined in Example 1. That is, $\int \phi(X_r) \, d\mu$ is a monotonic generalized sequence for each $\phi \in \mathcal{K}_0$. Since $\int |X_n| \, d\mu$ is bounded, so is $\int \phi(X_n) \, d\mu$, for each $\phi \in \mathcal{K}_0$. Hence, since $\int \phi(X_r) \, d\mu$ is monotonic, it is also bounded, for each $\phi \in \mathcal{K}_0$. As usual a
bounded, monotonic generalized sequence must converge. By the corollary to Theorem 1, $X_n$ must converge almost everywhere on $\Omega$ to a finite or infinite limit. The boundedness of $E(|X_n|)$ shows that the limit is finite almost everywhere, so Theorem 2 is proved.

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