ABSTRACT. Let \((\mathcal{B}, \ll)\) be a Banach lattice, and \((a, b)\) be an open interval on the real line. A function \(F: (a, b) \to \mathcal{B}\) is defined to be weakly convex if there exists a nonnegative nondecreasing continuous function \(G: (a, b) \to \mathcal{B}\) such that \(p[F(S)] + tp[G(s)] \leq p[F(s + t)]\), whenever \(s\) and \(s + t\) are in \((a, b)\) for each positive linear functional \(p\) on \(\mathcal{B}\). A representation theorem is proved as follows: If \(F\) is weakly convex on \((a, b)\) and is bounded on an interval contained in \((a, b)\), then 
\[
\int_{a + \epsilon}^{x} G(s) \, dm = F(x) - F(a + \epsilon),
\]
where \(\int_{a + \epsilon}^{x} G(s) \, dm\) is the Bochner integral of \(G\) on \([a + \epsilon, x]\) with \(0 < \epsilon\) and \(a + \epsilon < x < b\).

1. Introduction. Using a few facts in [1, pp. 91, 94, 95], an equivalent definition of convexity for continuous real-valued functions \(f\) can be formulated as follows: \(f\) is convex on an open interval \((a, b)\) if there exists a nondecreasing function \(g\) on \((a, b)\) such that \(f(s) + tg(s) \leq f(s + t)\), whenever \(s\) and \(s + t\) are in \((a, b)\). Based on this equivalent definition, in [2], the first author gave a definition of weak convexity for functions \(H: (a, b) \to \mathfrak{A}\), where \(\mathfrak{A}\) is a real commutative algebra closed in the strong topology of \(\mathfrak{H}(V, V)\), the space of all bounded hermitian operators on a Hilbert space \(V\). The results proved in [2] rely on the fact that \(\mathfrak{A}\) is a Dedekind complete lattice. Since \(\mathfrak{A}\) is also a Banach lattice in the sense given in [3, p. 366], a generalization of [2] to Banach lattices can be given; such is the purpose of this paper.

2. The weak convexity. A real Banach space \(\mathcal{B}\), with norm \(\| \cdot \|\), is called a Banach lattice if \(\mathcal{B}\) is a vector lattice under a partial ordering \(\ll\) such that \(|\beta| \ll |\alpha|\) implies \(\|\beta\| \leq \|\alpha\|\) for each \(\alpha\) and \(\beta\) in \(\mathcal{B}\).

Definition 1. Let \((a, b)\) be an open interval of the real line, and \(\theta\) be the zero vector in a Banach lattice \(\mathcal{B}\). A function \(F: (a, b) \to \mathcal{B}\) is weakly convex on \((a, b)\) if there exists a nondecreasing (w.r.t. \(\gg\)) and continuous

(w.r.t. \(\| \cdot \|\)) function \(G: (a, b) \to \mathbb{B}\) such that \(G(s) \gg \theta, s \in (a, b)\), and \(p[F(s)] + tp[G(s)] \leq p[F(s + t)],\) whenever \(s\) and \(s + t\) are in \((a, b)\) for each positive linear functional \(p\) on \(\mathbb{B}\).

**Theorem.** Suppose that \(F: (a, b) \to \mathbb{B}\) is weakly convex on \((a, b)\). If \(F\) is bounded on an interval \(J \subset (a, b)\) in the sense that there exists a \(K\) in \(\mathbb{B}\) such that \(\|F(t)\| = F^+(t) + F^-(t) \ll K\) for each \(t\) in \(J\), then

\[
(B) \int_{a + \epsilon}^x G(s) \, dm = F(x) - F(a + \epsilon),
\]
where \(\epsilon > 0\) and \(a + \epsilon \leq x < b\), and \((B)\int_{a + \epsilon}^x G(s) \, dm\) is the Bochner integral of \(G\) on \([a + \epsilon, x]\) with respect to the Lebesgue measure \(m\).

**Proof.** It is known \([3, p. 368]\) that order-boundedness and metric-boundedness are equivalent for linear functionals on a Banach lattice. Thus the space \(\mathbb{B}^b\) of all order-bounded linear functionals on the vector lattice \(\mathbb{B}\) is the same as the space \(\mathbb{B}^*\) of all norm continuous linear functionals on the Banach space \(\mathbb{B}\).

Let \(q\) be an element in \(\mathbb{B}^b = \mathbb{B}^*\). Then by the well-known Riesz decomposition theorem \([5, p. 68]\), \(q = q^+ - q^-\), where \(q^+\) and \(q^-\) are positive linear functionals on \(\mathbb{B}\). The weak convexity of \(F\) now implies that

\[
(q^+)(F(s)) + tq^+(G(s)) \leq q^+(F(s + t)), \quad s, s + t \in (a, b).
\]

The real-valued function \(q^+ \circ F\) is bounded on \(J\), since

\[
|q^+(F(s))| = |q^+(F^+(s)) - q^+(F^-(s))| \leq 2q^+(K)
\]
for each \(s\) in \(J\). To simplify notations, let \(q^+ \circ F = f, q^+ \circ G = g\), and rewrite (1) as follows:

(2) \(f(s) + tg(s) \leq f(s + t)\) whenever \(s\) and \(s + t \in (a, b)\); or

(3) \(f(s) - tg(s) \leq f(s - t)\) whenever \(s\) and \(s - t \in (a, b)\).

It follows from (2) and (3) that

(4) \(2f(s) \leq f(s + t) + f(s - t)\) whenever \(s + t\) and \(s - t \in (a, b)\).

This is equivalent to \(f(\frac{1}{2}s + \frac{1}{2}t) \leq \frac{1}{2}f(s) + \frac{1}{2}f(t)\) whenever \(s\) and \(t\) are in \((a, b)\). Since \(f\) is convex and bounded on \(J\), \(f\) is continuous on \((a, b)\) \([1, p. 91]\). From (2) and (3) it follows that for positive small \(\delta, f(s - \delta) + \delta g(s - \delta) \leq f(s)\) and \(f(s) - \delta g(s) \leq f(s - \delta)\). Therefore, \(g\) is nondecreasing on \((a, b)\).

For positive small \(t\), rewrite (2) as \(g(s) \leq (f(s + t) - f(s))/t\) and integrate both sides of this inequality over \([a + \epsilon, x]\), \(\epsilon > 0, a + \epsilon \leq x < b\), to obtain
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(5) \[
\int_{a+\epsilon}^{x} g(s) \, ds \leq \frac{\int_{a+\epsilon}^{x} (f(s) + t) \, ds - \int_{a+\epsilon}^{x} f(s) \, ds}{t}.
\]

Clearly, the reverse inequality holds for negative small \( t \) as follows:

(6) \[
\int_{a+\epsilon}^{x} g(s) \, ds \geq \frac{\int_{a+\epsilon}^{x} (f(s) + t) \, ds - \int_{a+\epsilon}^{x} f(s) \, ds}{t}.
\]

Passing to the limit as \( t \to 0 \), (5) and (6) yield

\[
\int_{a+\epsilon}^{x} g(s) \, ds = f(x) - f(a + \epsilon),
\]

i.e.

\[
\int_{a+\epsilon}^{x} q^+[G(s)] \, ds = q^+[F(x)] - q^+[F(a + \epsilon)].
\]

Similarly,

\[
\int_{a+\epsilon}^{x} q^-[G(s)] \, ds = q^-[F(x)] - q^-[F(a + \epsilon)].
\]

Thus for each bounded linear functional \( q \in \mathcal{B}^* \),

\[
\int_{a+\epsilon}^{x} q[G(s)] \, ds = \int_{a+\epsilon}^{x} (q^+ - q^-)[G(s)] \, ds
\]

(7) \[
= (q^+ - q^-)[F(x)] - (q^+ - q^-)[F(a + \epsilon)]
\]

\[
= q[F(x)] - q[F(a + \epsilon)].
\]

From (7) it is clear that \( G(s) \) is weakly Lebesgue integrable on \([a + \epsilon, x]\). Moreover, \( G(s) \) is strongly Lebesgue measurable on \([a + \epsilon, x]\), since \( G([a + \epsilon, x]) \), the range of \( G \), is separable in \( \mathcal{B} \) because of the continuity of \( G \) [4, p. 131]. The Bochner integral of \( G \) exists if the strongly Lebesgue measurable function \( G(s) \) has the property that \( \|G(s)\| \) is Lebesgue integrable [4, p. 133]. By assumption, with respect to \( \ll \), \( G \) is nondecreasing on \([a + \epsilon, x]\) with \( \theta \ll G(a + \epsilon) \ll G(x) \) in the Banach lattice \( \mathcal{B} \). Thus \( \|G(s)\| \) is nondecreasing on \([a + \epsilon, x]\) and, therefore, Lebesgue integrable. Let \( (B) \int_{a+\epsilon}^{x} G(s) \, dm \) denote the Bochner integral of \( G \) over \([a + \epsilon, x]\); then

(8) \[
(B) \int_{a+\epsilon}^{x} G(s) \, dm = F(x) - F(a + \epsilon)
\]

by (7) and the Hahn-Banach theorem. This completes the proof of the theorem.

3. The strong convexity. The notion of strong convexity for functions

\( F: (a, b) \to \mathcal{B} \) can be formulated as follows:

Definition 2. \( F: (a, b) \to \mathcal{B} \) is strongly convex on \((a, b)\) if there exists
a nondecreasing continuous function $G: (a, b) \rightarrow \mathbb{R}$ such that $G(t) \gg \theta$ for each $t$ in $(a, b)$ and $F(s) + tG(s) \ll F(s + t)$, whenever $s$ and $s + t$ are in $(a, b)$.

Clearly, the strong convexity implies the weak convexity. The validity of the converse statement is an open question. One natural way to proceed is to check whether or not $F(x) = F(a + \epsilon) + (B) \int_{a+\epsilon}^{x} G(s) \, dm$ satisfies $F(s) + tG(s) \ll F(s + t)$. The difficulty is that we do not know that the continuity and monotonicity of $G$ imply $(B) \int_{x}^{x+y} G(s) \, dm \gg yG(x)$.

Added in proof. The problem specifically mentioned here has been solved positively in [6].

REFERENCES


