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CONVEXITY OF VECTOR-VALUED FUNCTIONS

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ABSTRACT. Let $(\mathfrak{B}, \ll)$ be a Banach lattice, and (a, b) be an open interval on the real line. A function $F: (a, b) \to \mathfrak{B}$ is defined to be weakly convex if there exists a nonnegative nondecreasing continuous function $G: (a, b) \to \mathfrak{B}$ such that $p[F(S)] + tp[G(s)] \le p[F(s + t)]$, whenever s and $s + t$ are in (a, b) for each positive linear functional p on $\mathfrak{B}$. A representation theorem is proved as follows: If F is weakly convex on (a, b) and is bounded on an interval contained in (a, b), then $(B)\int_{a+\epsilon}^{x} G(s)\,dm = F(x) - F(a + \epsilon)$, where $(B)\int_{a+\epsilon}^{x} G(s)\,dm$ is the Bochner integral of G on $[a + \epsilon, x]$ with $0 < \epsilon$ and $a + \epsilon < x < b$.

1. **Introduction.** Using a few facts in [1, pp. 91, 94, 95], an equivalent definition of convexity for continuous real-valued functions f can be formulated as follows: f is convex on an open interval (a, b) if there exists a nondecreasing function g on (a, b) such that $f(s) + tg(s) \le f(s + t)$, whenever s and $s + t$ are in (a, b). Based on this equivalent definition, in [2], the first author gave a definition of weak convexity for functions $H: (a, b) \to \mathfrak{A}$, where $\mathfrak{A}$ is a real commutative algebra closed in the strong topology of $\mathfrak{H}(V, V)$, the space of all bounded hermitian operators on a Hilbert space V. The results proved in [2] rely on the fact that $\mathfrak{A}$ is a Dedekind complete lattice. Since $\mathfrak{A}$ is also a Banach lattice in the sense given in [3, p. 366], a generalization of [2] to Banach lattices can be given; such is the purpose of this paper.

2. **The weak convexity.** A real Banach space $\mathfrak{B}$, with norm $\|\cdot\|$, is called a Banach lattice if $\mathfrak{B}$ is a vector lattice under a partial ordering $\ll$ such that $|\beta| \ll |\alpha|$ implies $\|\beta\| \le \|\alpha\|$ for each α and β in $\mathfrak{B}$.

Definition 1. Let (a, b) be an open interval of the real line, and θ be the zero vector in a Banach lattice $\mathfrak{B}$. A function $F: (a, b) \to \mathfrak{B}$ is weakly convex on (a, b) if there exists a nondecreasing (w.r.t. $\gg$) and continuous

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(w.r.t. $\|\cdot\|$) function $G: (a, b) \to \mathfrak{B}$ such that $G(s) \gg \theta$, $s \in (a, b)$, and $p[F(s)] + tp[G(s)] \le p[F(s+t)]$, whenever s and $s+t$ are in (a, b) for each positive linear functional p on $\mathfrak{B}$.

Theorem. *Suppose that $F: (a, b) \to \mathfrak{B}$ is weakly convex on (a, b). If F is bounded on an interval $J \subset (a, b)$ in the sense that there exists a K in $\mathfrak{B}$ such that $|F(t)| = F^+(t) + F^-(t) \ll K$ for each t in J, then*

$$(B) \int_{a+\epsilon}^{x} G(s)\, dm = F(x) - F(a+\epsilon),$$

where $\epsilon > 0$ and $a + \epsilon \le x < b$, and $(B) \int_{a+\epsilon}^{x} G(s)\, dm$ is the Bochner integral of G on $[a+\epsilon, x]$ with respect to the Lebesgue measure m.

Proof. It is known [3, p. 368] that order-boundedness and metric-boundedness are equivalent for linear functionals on a Banach lattice. Thus the space $\mathfrak{B}^b$ of all order-bounded linear functionals on the vector lattice $\mathfrak{B}$ is the same as the space $\mathfrak{B}^*$ of all norm continuous linear functionals on the Banach space $\mathfrak{B}$.

Let q be an element in $\mathfrak{B}^b = \mathfrak{B}^*$. Then by the well-known Riesz decomposition theorem [5, p. 68], $q = q^+ - q^-$, where q^+ and q^- are positive linear functionals on $\mathfrak{B}$. The weak convexity of F now implies that

$$(1) \qquad q^+[F(s)] + tq^+[G(s)] \le q^+[F(s+t)], \qquad s,\ s+t \in (a, b).$$

The real-valued function $q^+ \circ F$ is bounded on J, since

$$|q^+[F(s)]| = |q^+[F^+(s)] - q^+[F^-(s)]| \le 2q^+(K)$$

for each s in J. To simplify notations, let $q^+ \circ F = f$, $q^+ \circ G = g$, and rewrite (1) as follows:

$$(2) \qquad f(s) + tg(s) \le f(s+t) \quad \text{whenever } s \text{ and } s+t \in (a, b); \text{ or}$$

$$(3) \qquad f(s) - tg(s) \le f(s-t) \quad \text{whenever } s \text{ and } s-t \in (a, b).$$

It follows from (2) and (3) that

$$(4) \quad 2f(s) \le f(s+t) + f(s-t) \quad \text{whenever } s+t \text{ and } s-t \in (a, b).$$

This is equivalent to $f(\frac{1}{2}s + \frac{1}{2}t) \le \frac{1}{2}f(s) + \frac{1}{2}f(t)$ whenever s and t are in (a, b). Since f is convex and bounded on J, f is continuous on (a, b) [1, p. 91]. From (2) and (3) it follows that for positive small δ, $f(s-\delta) + \delta g(s-\delta) \le f(s)$ and $f(s) - \delta g(s) \le f(s-\delta)$. Therefore, g is nondecreasing on (a, b).

For positive small t, rewrite (2) as $g(s) \le (f(s+t) - f(s))/t$ and integrate both sides of this inequality over $[a+\epsilon, x]$, $\epsilon > 0$, $a + \epsilon \le x < b$, to obtain

$$\int_{a+\epsilon}^{x} g(s)\,ds \leq \frac{\int_{a+\epsilon}^{x} f(s+t)\,ds - \int_{a+\epsilon}^{x} f(s)\,ds}{t}. \tag{5}$$

Clearly, the reverse inequality holds for negative small t as follows:

$$\int_{a+\epsilon}^{x} g(s)\,ds \geq \frac{\int_{a+\epsilon}^{x} f(s+t)\,ds - \int_{a+\epsilon}^{x} f(s)\,ds}{t}. \tag{6}$$

Passing to the limit as $t \to 0$, (5) and (6) yield

$$\int_{a+\epsilon}^{x} g(s)\,ds = f(x) - f(a+\epsilon),$$

i.e.

$$\int_{a+\epsilon}^{x} q^{+}[G(s)]\,ds = q^{+}[F(x)] - q^{+}[F(a+\epsilon)].$$

Similarly,

$$\int_{a+\epsilon}^{x} q^{-}[G(s)]\,ds = q^{-}[F(x)] - q^{-}[F(a+\epsilon)].$$

Thus for each bounded linear functional $q \in \mathfrak{B}^*$,

$$\begin{aligned} \int_{a+\epsilon}^{x} q[G(s)]\,ds &= \int_{a+\epsilon}^{x} (q^{+} - q^{-})[G(s)]\,ds \\ &= (q^{+} - q^{-})[F(x)] - (q^{+} - q^{-})[F(a+\epsilon)] \qquad (7) \\ &= q[F(x)] - q[F(a+\epsilon)]. \end{aligned}$$

From (7) it is clear that $G(s)$ is weakly Lebesgue integrable on $[a+\epsilon, x]$. Moreover, $G(s)$ is strongly Lebesgue measurable on $[a+\epsilon, x]$, since $G[a+\epsilon, x]$, the range of G, is separable in $\mathfrak{B}$ because of the continuity of G [4, p. 131]. The Bochner integral of G exists if the strongly Lebesgue measurable function $G(s)$ has the property that $\|G(s)\|$ is Lebesgue integrable [4, p. 133]. By assumption, with respect to $\ll$, G is nondecreasing on $[a+\epsilon, x]$ with $\theta \ll G(a+\epsilon) \ll G(x)$ in the Banach lattice $\mathfrak{B}$. Thus $\|G(s)\|$ is nondecreasing on $[a+\epsilon, x]$ and, therefore, Lebesgue integrable. Let $(B)\int_{a+\epsilon}^{x} G(s)\,dm$ denote the Bochner integral of G over $[a+\epsilon, x]$; then

$$(B)\int_{a+\epsilon}^{x} G(s)\,dm = F(x) - F(a+\epsilon)$$

by (7) and the Hahn-Banach theorem. This completes the proof of the theorem.

3. The strong convexity. The notion of strong convexity for functions $F\colon (a, b) \to \mathfrak{B}$ can be formulated as follows:

Definition 2. $F\colon (a, b) \to \mathfrak{B}$ is strongly convex on (a, b) if there exists a nondecreasing continuous function G: $(a, b) \to \mathfrak{B}$ such that $G(t) \gg \theta$ for each t in (a, b) and $F(s) + tG(s) \ll F(s+t)$, whenever s and $s+t$ are in (a, b).

Clearly, the strong convexity implies the weak convexity. The validity of the converse statement is an open question. One natural way to proceed is to check whether or not $F(x) = F(a+\epsilon) + (B)\int_{a+\epsilon}^{x} G(s)\,dm$ satisfies $F(s) + tG(s) \ll F(s+t)$. The difficulty is that we do not know that the continuity and monotonicity of G imply $(B)\int_{x}^{x+y} G(s)\,dm \gg yG(x)$.

Added in proof. The problem specifically mentioned here has been solved positively in [6].

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