ON THE SET OF EXTREME POINTS OF A CONVEX BODY

JAMES B. COLLIER

ABSTRACT. We prove the following: Given a subset \( X \) of a compact 0-dimensional metric space \( Z \) and an integer \( d \geq 3 \), there is a homeomorphism of \( Z \) into the boundary of a convex body \( C \) in \( E^d \) mapping \( X \) onto the set of extreme points of \( C \) if and only if \( X \) is a \( G_\delta \) set with at least \( d + 1 \) points.

A convex body is any closed bounded convex set with nonempty interior. It is well known that the set of extreme points of a convex body in \( E^d \) is a \( G_\delta \) set with at least \( d + 1 \) points. We may consider as a partial converse to this the fact that for each integer \( n \geq d + 1 \) there is a convex body in \( E^d \) with exactly \( n \) extreme points. The purpose of this note is to prove a more general converse. We denote the set of extreme points of a convex set \( C \) by \( \text{ext} \ C \). Our main result is

**Theorem 1.** Let \( X \) be a subset of a compact 0-dimensional metric space \( Z \), and let \( d \) be an integer with \( d \geq 3 \). There is a homeomorphism of \( Z \) into the boundary of a convex body \( C \) in \( E^d \) mapping \( X \) onto \( \text{ext} \ C \) if and only if \( X \) is a \( G_\delta \) set with at least \( d + 1 \) points.

The boundary of any convex body in \( E^d \) is homeomorphic to the \((d - 1)\)-sphere, \( S^{d-1} \). For each positive integer \( d \) we define \( \chi_d \) to be the family of all subsets \( X \) of \( S^{d-1} \) for which there is a homeomorphism of \( S^{d-1} \) onto the boundary of a convex body \( C \) in \( E^d \) mapping \( X \) onto \( \text{ext} \ C \). The family we call \( \chi_3 \) was first defined in [3] and the question raised of finding a characterization of that family. It is known that each member of \( \chi_d \) is a \( G_\delta \) set with at least \( d + 1 \) points and that each closed subset of \( S^{d-1} \) with at least \( d + 1 \) points is a member of \( \chi_d \). However, no proper open subset of \( S^{d-1} \) is in \( \chi_d \). Using Theorem 1 we may further state

**Corollary 1.** Any \( G_\delta \) subset of \( S^2 \) with at least four points and whose closure is 0-dimensional is a member of \( \chi_3 \).

Received by the editors October 30, 1973 and, in revised form, December 17, 1973.

Corollary 2. For $d \geq 3$ any subset of $S^{d-1}$ with at least $d + 1$ points and whose closure is countable is a member of $\mathcal{X}_d$.

These follow easily from results in [4, p. 532] and [2, p. 456], respectively, on extending homeomorphisms.

We proceed to the proof of Theorem 1. The closure, the boundary, and the convex hull of a subset $A$ of $E^d$ will be denoted by $\text{cl} A$, $\text{bd} A$, and $\text{conv} A$, respectively. For any compact subset $K$ of the real line $\mathbb{R}$, $\mathbb{R}\setminus K$ is the disjoint union of countably many open intervals. Let $e(K)$ be the set of endpoints of these intervals; then $e(K)$ is a countable subset of $K$.

**Lemma 1.** Let $Y$ be a compact 0-dimensional subset of $\mathbb{R}$, and $X$ a $G_\delta$ subset of $Y$ which contains $e(Y)$. Then there is a continuous function $g: Y \to \mathbb{R}$ such that $X \cap g^{-1}(s)$ is the set of extreme points of $\text{conv} \ g^{-1}(s)$ for each $s \in g[Y]$.

**Proof.** We construct a sequence $U_0, U_1, \ldots$ of open (and also closed) coverings of $Y$ by pairwise disjoint sets so that for each $n \geq 0$, $U_{n+1}$ is a refinement of $U_n$. For each $U$ in $U_n$ we shall specify a compact subset $c_n(U)$ of $U$.

Since $X$ is a $G_\delta$ set, $Y\setminus X$ is the countable union of a nested chain $K_1 \subseteq K_2 \subseteq \ldots$ of compact sets. We may choose $U_0 = \{Y\}$ and $c_0(Y) = \emptyset$ and suppose that $U_{n-1}$ and $c_{n-1}$ have been defined for some $n \geq 1$. To complete the inductive construction, we need only specify for an arbitrary $U \in U_{n-1}$ the members $V_0, \ldots, V_k$ of $U_n$ which are subsets of $U$ and define $c_n(U)$ for $i = 0, \ldots, k$. Since $U$ is compact and 0-dimensional, there is a finite open covering $\mathcal{U}$ of $U$ by pairwise disjoint sets each of diameter $\leq 1/n$. Let $V_0$ be the union of those members of $\mathcal{U}$ which intersect $c_{n-1}(U)$ and let $V_1, \ldots, V_k$ be an enumeration of the remaining sets in $\mathcal{U}$. Choose $c_n(V_0) = c_{n-1}(U)$. By hypothesis the extreme points $a, b$ of $\text{conv} V_i$ are in $X$. We define $c_n(V_i) = \{a, b\} \cup (K_n \cap V_i)$ for $i = 1, \ldots, k$. Hence the extreme points of $\text{conv} c_n(V_i)$ are $a, b$ and these are the only points of $c_n(V_i)$ which lie in $X$.

Consider each $U_n$ to be a topological space with the discrete topology. Then there is a continuous mapping $f_n: U_n \to U_{n-1}$, $n \geq 1$, defined by $U \subseteq f_n(U)$, and there is a continuous mapping $g_n: Y \to U_n$ defined by $y \in g_n(y)$. Thus $\{U_n, f_n\}$ is an inverse limit sequence. Let $U_\infty$ be the inverse limit space of $\{U_n, f_n\}$, and let $g_\infty: Y \to U_\infty$ be the continuous mapping induced by $g_n: Y \to U_n$. Since $Y$ is compact and $U_n$ is discrete for each $n \geq 0$, $U_\infty$ is compact and 0-dimensional; hence there is a homeomorphism.
We define $g = h \circ g_\infty$. From the construction of $g_\infty$ it is evident that for $x, y \in Y$, $g(x) = g(y)$ if and only if either $x = y$ or $x, y \in c_n(U)$ for some $U \in \mathcal{U}_n$ and $n \geq 0$. Since each point in $Y \setminus X$ lies in some $c_n(U)$ but is not an extreme point of $\text{conv} c_n(U)$, $g$ has the desired property. □

The next lemma may also be proved using inverse limit spaces. However, we omit the details since the method is similar to that used in [1, §§2—15].

**Lemma 2.** Let $Y$ be a compact, 0-dimensional metric space and $X$ a countable dense subset of $Y$. Then there is a homeomorphism $h: Y \to \mathbb{R}$ such that $e(h[Y]) = e[X]$.

**Proof of Theorem 1.** One direction is immediate. To prove the other direction, suppose $X$ is an infinite $G_\delta$ subset of $Z$. Lemma 2 implies that we may assume $\text{cl} X = Y \subseteq \mathbb{R}$ with $e(Y) \subseteq X$. Let $g: Y \to \mathbb{R}$ be as in Lemma 1, and define $G: Y \to E^3$ by $x \mapsto (x, g(x), g^2(x))$. This is a homeomorphism of $Y$ into the graph of the convex function on $E^2$ given by $(x, y) \mapsto y^2$. Let $C = \text{conv} G[Y]$, then $\text{ext} C = G[X]$. Since we may extend $G$ to a homeomorphism $h: Z \to \text{bd} C$, we have established Theorem 1 for $d = 3$.

Assume the theorem is true for $d = n$, and express $Z$ as the disjoint union of closed subsets $Z_1, Z_2$ so that $X_1 = X \cap Z_1$ is infinite, $i = 1, 2$. By assumption there is a convex body $C_i$ in $E^n$ and a homeomorphism $h_i: Z_i \to \text{bd} C_i$ with $h_i[X_i] = \text{ext} C_i$. We may consider $C_1, C_2$ to lie in distinct parallel hyperplanes in $E^{n+1}$. Let $C = \text{conv}(C_1 \cup C_2)$ and define the homeomorphism $h: Z \to \text{bd} C$ by $h/Z_i = h_i$, $i = 1, 2$. Then $C$ is a convex body in $E^{n+1}$ and $h[X] = \text{ext} C$. □

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF SOUTHERN CALIFORNIA, LOS ANGELES, CALIFORNIA 90007