COMMUTATIVE REGULAR RINGS WITHOUT
PRIME MODEL EXTENSIONS

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ABSTRACT. It is known that the theory $K$ of commutative regular rings with identity has a model completion $K'$. We show that there exists a countable model of $K$ which has no prime extension to a model of $K'$.

If $K$ and $K'$ are theories in a first order language $L$, then $K'$ is said to be a model completion of $K$ if $K'$ extends $K$, every model of $K$ can be embedded in a model of $K'$, and for any model $A$ of $K$ and models $B_1$, $B_2$ of $K'$ extending $A$, we have $(B_1 a)_{a \in A} \equiv (B_2 a)_{a \in A}$, i.e. $B_1$ and $B_2$ are elementarily equivalent in a language which has constants for the elements of $A$. If a theory $K$ has a model completion $K'$, then the models of $K'$ can reasonably be regarded as the "algebraically closed" models of $K$; for example, the theory of algebraically closed fields is the model completion of the theory of fields. It was shown in [3] that the theory $K$ of commutative regular rings with identity (formulated in the usual language $L$ for rings with identity) has a model completion. We recall that a commutative ring $R$ with identity is said to be regular (in the sense of von Neumann) if for any element $a \in R$ there exists $b \in R$ such that $a^2 b = a$. (A good reference is Lambek [2].) The model completion $K'$ is given by the following axioms:

(i) the axioms of commutative regular rings with identity;
(ii) an axiom stating that there are no minimal idempotents, i.e.

$$\forall x(x^2 = x \land x \neq 0 \rightarrow \exists y(y^2 = y \land y \neq 0 \land y \neq x \land yx = y));$$

(iii) a set of axioms stating that every monic polynomial has a root.

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We will also consider the theory $K^*$ obtained by deleting axiom (ii) from $K'$.

If $B$ is a model of a theory $T$, and $A$ is a substructure of $B$, then $B$ is called a prime model extension (for $T$) of $A$, if for every model $C$ of $T$ extending $A$, there exists an embedding $f: B \to C$ which is the identity on $A$. Of course if $T$ is model-complete (as in $K'$), then "embedding" can be replaced by "elementary embedding". In this paper we consider the question of whether over every commutative regular ring $A$ there exists a prime model extension for $K'$; we also consider the same question for $K^*$. In both cases the answer is negative.

We begin by recalling some model theoretic preliminaries in the setting of commutative regular rings. Let $R$ be a model of $K$; we expand $L$ to the language $L(R)$ by adding a new constant symbol $a_1$ for every element $a \in R$. The diagram $D(R)$ of $R$ is the set of all polynomial equations and inequalities involving the constants $a$ which hold in $R$ when $a_1$ is interpreted as $a$. Every model of the theory $K' \cup D(R)$ is (up to an isomorphism) an extension of $R$.

Let $F(R)$ be the set of formulas in $L(R)$ with one free variable $x$. For $\phi, \psi$, in $F(R)$ we set $\phi \sim \psi$, if and only if $K' \cup D(R) \vdash \phi \leftrightarrow \psi$. This gives us an equivalence relation on $F(R)$; we denote the equivalence class of $\phi$ by $[\phi]$. The equivalence classes form a Boolean algebra $B(R)$ (called the Lindenbaum algebra of $K' \cup D(R)$) under the operations $[\phi] + [\psi] = [\phi \lor \psi], [\phi]^* = [\neg \phi]$. We call the elements of the Stone space $S(R)$ of $B(R)$ $1$-types over $R$. Notice that a point $p \in S(R)$ is isolated in the Stone topology if and only if there is a formula $\phi$ such that $K' \cup D(R) \vdash \phi \iff \psi$ for every $\psi$ such that $[\psi] \in p$, and $[\phi] \neq 0$ in $B(R)$. Such a formula $\phi$ is called a generator for $p$.

Since $K' \cup D(R)$ is complete, it is clear that if $p \in S(R)$ is isolated, then every model $A$ of $K' \cup D(R)$ realizes $p$, i.e. there exists $a \in A$ such that $[[\phi]] : A \models \phi(a) = p$.

We shall present our results within the framework of finite forcing relative to $K \cup D(R)$ (for $R$ countable). The fundamental papers on finite forcing in model theory are [1] and [4]. We expand $L(R)$ to $L(R, C)$ by adding a countable set $C$ of new constant symbols. In our setting, a forcing condition $q(a_1, \ldots, a_n, c_1, \ldots, c_m)$ is a finite set of polynomial equations and inequalities in the language $L(R, C)$ which is consistent with $K \cup D(R)$, i.e. such that there exists a model $A$ of $K$ extending $R$ and $c_1, \ldots, c_m$ in $A$ such that $A$ satisfies all the statements in $q$ when each $a_i$ is interpreted as $a_i$, and each $c_i$ is interpreted as $c_i$. If $q$ is a condition and $\phi$ is a...
sentence in $L(R, C)$, then the notion "$q$ forces $\phi$" is defined by induction on the structure of $\phi$, as follows:

(i) If $\phi$ is an equation, $q$ forces $\phi$ if and only if $\phi \in q$;
(ii) $q$ forces $\phi \land \psi$ if and only if $q$ forces $\phi$ and $q$ forces $\psi$;
(iii) $q$ forces $\phi \lor \psi$ if and only if $q$ forces $\phi$ or $q$ forces $\psi$ or both;
(iv) $q$ forces $\exists x \, \phi(x)$ if and only if for some $c \in C$, $q$ forces $\phi(c)$;
(v) $q$ forces $\forall \phi$ if and only if for no condition $q'$ extending $q$ is it the case that $q'$ forces $\phi$.

A sequence $\{q_i\}_{i=1}^{\infty}$ of conditions is complete if for each sentence $\phi$ of $L(R, C)$ there is some $q_i$ which forces either $\phi$ or $\neg \phi$. A complete sequence determines in a canonical way (see [1] or [4]) a ring $A$ which contains $R$. Every element of $A$ is named by some $c$ in such a way that all the statements in any $q_i$ hold in $A$ (when $a$ is interpreted as $a$ for $a \in R$), and any equation which holds in $A$ is in some $q_i$. $A$ is called finitely generic for $K \cup D(R)$. Since $K \cup D(R)$ has a model completion (namely $K'$ $\cup$ $D(R)$), this implies [1] that $A$ is a model of $K' \cup D(R)$; in particular $A$ is a model of $K'$.

We can now prove

**Theorem 1.** There exists a countable model $R$ of $K$ which has no prime extension to a model of $K'$. Moreover, $R$ can be chosen so that all the isolated points in $S(R)$ are realized in $R$, i.e. have a generator $x = r$ for some $r \in R$. In particular the isolated points are not dense in $S(R)$.

**Corollary.** $K'$ is not quasi-totally transcendental. (For this notion see [5].)

**Remark.** It is easy to see that $K'$ is $\kappa$-unstable for all infinite cardinals $\kappa$.

**Proof of Theorem 1.** Let $R'$ be the ring of all locally constant functions from the Cantor space $X$ into $\overline{Q}$, an algebraic closure of the rationals $Q$. (A function $f$ on $X$ is locally constant if for every $x \in X$ there exists a neighborhood $U$ of $x$ on which $f$ is constant.) $R'$ is a model of $K'$ and $R'$ is countable. For, notice that any locally constant function $f$ on $X$ determines a partition of $X$ into finitely many clopen sets $P_i$ such that $f$ is constant on each $P_i$. Since $\overline{Q}$ is countable, any such partition is determined in this way by only countably many $f$'s; and there are only countably many such partitions of $X$.

Pick a point $x_0 \in X$. Let $R \subset R'$ consist of all the elements $f \in R'$ such that $f(x_0) \in Q$. It is easy to see that $R$ is a regular ring.
Now suppose \( R \) has a prime extension \( M \) to a model of \( K' \). We can assume \( R \subseteq M \subseteq R' \), and then we know that \( M \) is an elementary substructure of \( R' \). Since \( R \) is not a model of \( K' \), we can pick \( d \in M - R \).

Let \( p \) be the point of \( S(R) \) realized by \( d \) in \( M \). We will show that there exists a model of \( K' \cup D(R) \), no element of which realizes \( p \); it follows that \( M \) cannot be embedded in this model over \( R \), so \( M \) is not prime.

Let \( C \), as before, be a countable set of new constants not in \( L(R) \), and let \( \{ \phi_i \}_{i=1}^\infty \) be an enumeration of the sentences in \( L(R, C) \). We will in a moment define a complete sequence \( \{ q_i \} \) of forcing conditions, but first we state the following

**Lemma.** Let \( q \) be a condition. Let \( c_1, \ldots, c_n \) be the elements of \( C \) mentioned in \( q \). Then there exist elements \( a_1, b_1, \ldots, a_n, b_n \) in \( R \) such that \( q \cup \{ b_i \neq c_i a_i \mid i = 1, \ldots, n \} \) is a condition, where \( a_i, b_i \in R \) are such that \( b_i = da_i \) in \( R' \).

We will prove the Lemma later.

Now define a sequence \( \{ q_i \} \) of conditions as follows: Let \( q_1 \) be a condition which forces either \( \phi_1 \) or \( \neg \phi_1 \). Let \( q_2 \) be an extension of \( q_1 \) obtained by using the Lemma. If \( i > 1 \) is odd, let \( q_i \) be an extension of \( q_{i-1} \) which forces either \( \phi_{(i+1)/2} \) or \( \neg \phi_{(i+1)/2} \). If \( i > 1 \) is even, let \( q_i \) be an extension of \( q_{i-1} \) obtained by using the Lemma. As indicated above, the complete sequence \( \{ q_i \} \) determines a model \( R'' \) of \( K' \) which extends \( R \), and every element of \( R'' \) is denoted by some \( c \in C \).

Suppose some element \( r \in R'' \) realizes \( p \), and that this element is denoted by \( c_i \). Then for some odd \( j \), \( c_i \) appears in \( q_j \). Thus there exist \( a_i, b_i \) in \( R \) such that \( b_i = da_i \) in \( R' \) and \( q_j \) contains the statement \( b_i \neq c_i a_i \). Since \( b_i \neq c_i a_i \) is in \( q_{j+1} \), \( b_i \neq xa_i \) is in the type realized by \( r \) in \( R'' \). But \( b_i = xa_i \) is in \( p \), since \( p \) is the type realized by \( d \) in \( R' \) and \( b_i = da_i \). Thus \( r \) does not realize \( p \) in \( R'' \). This proves the first statement of the theorem.

To prove the second statement, suppose there is an isolated point \( p \) in \( S(R) \) which is not realized in \( R \). Since \( p \) is isolated there is \( d \in R' - R \) which realizes \( p \). Then by the above, there exists a model of \( K' \cup D(R) \), no element of which realizes \( p \). Hence \( p \) is not isolated, a contradiction.

To see that the isolated points are not dense in \( S(R) \), observe that if \( \phi(x) \) is a formula in \( L(R) \) such that \( K' \cup D(R) \models \exists x \phi(x) \), but \( R \models \forall x \phi(x) \), then the neighborhood in \( S(R) \) determined by \( \phi(x) \) contains no isolated
points, since all the isolated points are realized in $R$. An example of such a formula is $x^2 = 2$.

This finishes the proof, modulo the Lemma.

In proving the Lemma we will work with idempotents of $R'$, and it will be helpful to make some preparatory remarks about them. An idempotent $e$ of $R'$ is an element $e$ in $R'$ whose values are everywhere either 0 or 1. We identify $e$ with the clopen subset of $X$ consisting of all points $x \in X$ such that $e(x) = 1$. This gives a 1-1 correspondence between the idempotents of $R'$ and the clopen subsets of $X$. Thus if $e, f$ are idempotents, we say $e \subseteq f$ if $e(x) = 1$ implies $f(x) = 1$ for all $x \in X$. Furthermore the correspondence makes it clear what we mean when we say that an element of $R'$ is constant on some idempotent.

Let $q$ be a condition, let $h_1, \ldots, h_m$ be all the elements of $R$ such that $h_i$ occurs in $q$, and let $c_1, \ldots, c_n$ be the elements of $C$ which occur in $q$. Let $\phi(x_1, \ldots, x_n)$ be the formula obtained by replacing $c_i$ by $x_i$ in the conjunction of the elements of $q$. Then since $q$ is a condition, the formula $\exists x_1 \cdots \exists x_n \phi$ holds in some model of $K' \cup D(R)$, and hence in all of them, since $K'$ is the model completion of $K$. In particular $R' \models \exists x_1 \cdots \exists x_n \phi$, so there exist elements $c_1, \ldots, c_n$ in $R'$ such that $R' \models \phi(c_1, \ldots, c_n)$ when $c_i$ is interpreted as $c_i$ and $h_i$ is interpreted as $h_i$.

$\phi(c_1, \ldots, c_n)$ is a conjunction of polynomial equations $\{p_{i_1} = p_{i_2}\}_{i=1}^s$ and inequations $\{p_{s+i,1} \neq p_{s+i,2}\}_{i=1}^t$ which hold in $R'$. For each $i$, $1 \leq i \leq t$, let $x_i \neq x_0$ be a point in $X$ at which $P_{s+i,1}$ holds in $R'$. Let $e$ be an idempotent of $R$ such that $e(x_0) = 1$ and $e(x_i) = 0$ for $1 \leq i \leq t$. (The fact that $e$ exists follows from the properties of $X$.) Let $f \subseteq e$ be an idempotent such that $f(x_0) = 1$ and each of $d, h_1, \ldots, h_m, c_1, \ldots, c_n$ is constant on $f$. Notice that since each $h_i \in R$, each $h_i$ has a constant rational value on $f$.

Let $d^*$ denote the constant irrational value of $d$ on $f$. Let $u, v$ be in $R$ be idempotents such that $u, v \subseteq f$, $u(x_0) = u(x_0) = 0$, $uv = 0$, and $u \neq 0$, $v \neq 0$. Let $x_1, x_2$ in $R$ be such that $y(x) = d^*$ for all $x \in u$ and $y(x) = 0$ for all $x \notin u$, $z(x) = d^*$ for all $x \in v$, $z(x) = 0$ for all $x \notin v$.

Since $d^*$ is irrational, there exists an automorphism $\rho$ of $R$ such that $\rho(d^*) \neq d^*$. Denote the constant values of the $h_i$ and $c_i$ on $f$ by $h_i^*, c_i^*$, respectively. Then each $h_i^* \in Q$, so $\rho(h_i^*) = h_i^*$ for each $i$. Let $c_i^*, 1 \leq i \leq n, be the element of $R'$ which has the constant value $\rho(c_i^*)$ on $u$ and the same values as $c_i$ on $1 - u$.

Let $J = \{i \in \{1, \ldots, n\}| c_i$ has the constant value $d^*$ on $f\}$. For $i \in J$ let $a_i = u$ and let $b_i = y$. For $i \notin J$ let $a_i = v$ and let $b_i = z$.
We claim that \( q \cup \{ \hat{b}_i \neq \hat{c}_i \, | \, 1 \leq i \leq n \} \) holds in \( R' \) when we interpret \( \hat{a}_i \) as \( a_i \), \( \hat{b}_i \) as \( b_i \), \( \hat{c}_i \) as \( c_i \), and \( \hat{c}_i' \) as \( c_i' \). The fact that the part in brackets holds follows immediately from the definition of the \( \hat{a}_i \), \( \hat{b}_i \), and \( \hat{c}_i \). The inequalities in \( q \) hold, because the \( c_i' \) agree with the \( c_i \) on \( x_1, \ldots, x_t \). The equalities hold, because the \( c_i' \) differ from the \( c_i \) only on \( u \), and on \( u \) the equalities hold for the \( c_i' \), because \( \rho \) is an automorphism of \( \mathcal{Q} \).

This completes the proof.

Remarks. (1) Using the model completeness of \( K' \), one can argue directly from the Lemma that the type of \( d \) is not principal; one can then fall back on the standard omitting types theorem to conclude that \( R \) has no prime extension to a model of \( K' \). However we feel that the presentation in terms of forcing is more intuitive and more nearly self-contained.

(2) In the general framework of [5], one considers theories which are model completions of universal theories.

In any commutative regular ring there exists for any element \( x \) a unique element \( f(x) \) such that \( x^2f(x) = x \) and \( f(x)^2x = f(x) \) (see [2]). If we enlarge our language \( L \) to \( L' \) by adding a 1-place function symbol \( f \), and write the axiom of regularity in the form

\[
\forall x(x^2f(x) = x \land f(x)^2x = f(x)),
\]

then, as is remarked in [3], \( K' \) is in \( L' \) the model completion of \( K \), and \( K \) in \( L' \) is a universal theory. Since the function \( f \) is definable in \( L \), it is easy to see that Theorem 1 holds for \( L' \) as well as for \( L \). Thus with respect to \( L' \), \( K' \) is a natural example of a model completion of a universal theory \( K \) which has a countable model \( R \) such that the isolated points are not dense in \( S(R) \).

Recall that \( K^* \) is \( K \) with axiom (ii) deleted.

**Theorem 2.** If \( R \) is the ring of Theorem 1, then \( R \) has no prime extension to a model of \( K^* \).

**Proof.** Suppose \( A \) is a prime extension of \( R \) to a model of \( K^* \). Then since \( R' \) (as defined in the proof of Theorem 1) is a model of \( K^* \), there exists an embedding \( f: A \rightarrow R' \) such that \( f|R \) is the identity map. Therefore \( A \) has the same idempotents as \( R \), since \( R' \) has the same idempotents as \( R \). Thus there are no minimal idempotents in \( A \), so \( A \) is in fact a model of \( K' \). Thus \( A \) is a prime extension of \( R \) to a model of \( K' \), contradicting Theorem 1.

We should point out that \( R' \) is a minimal extension of \( R \) to a model of
K' (and K*). That is, there is no model of K' (or K*) sitting strictly between $R$ and $R'$. For suppose $S$ is a model of K* and $R \subseteq S \subseteq R'$. Then for any $x \in X$, $\{f(x)|f \in S\} = \overline{Q}$. Now let $f \in R'$; we claim $f \in S$. To see this, we observe that if $x \in X$ there exists an idempotent $e_x$ containing $x$ and $f_x \in S$ such that $f_x$ and $f$ have the same constant value on $e_x$. It follows from the properties of $X$ that there exist disjoint idempotents $e_1$, $\cdots$, $e_n$ in $R$ and elements $f_1, \cdots, f_n$ in $S$ such that $\sum_{i=1}^{n} e_i = 1$, and $f_i e_i = f e_i$, $1 \leq i \leq n$. Then

$$f = \sum_{i=1}^{n} f e_i = \sum_{i=1}^{n} f_i e_i \in S.$$

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