A FORCING PROOF OF THE
KECHRIS-MOSCHOVAKIS CONSTRUCTIBILITY THEOREM

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ABSTRACT. We show, by forcing, that every subset of \( \mathbb{N} \) whose
codes form a \( \Sigma^1_2 \) set of reals must be constructible.

In [1], Kechris and Moschovakis proved the following theorem by a game-
theoretic argument and expressed doubt whether it could be proved by the
forcing techniques of Solovay [3].

Theorem (Kechris-Moschovakis). Let \( A \) be a set of countable ordinals
whose codes form a \( \Sigma^1_2 \) set of reals. Then \( A \) is constructible.

(For details of the coding of ordinals by reals, see [1].)
The purpose of this note is to prove this theorem by forcing.

Let \( A \) be as in the hypothesis of the theorem, and let \( P \) be a \( \Sigma^1_2 \) formula
such that, whenever a real \( \alpha \) codes an ordinal \( \sigma \),

\[
\sigma \in A \iff P(\alpha).
\]

We may suppose, without loss of generality, that the statement

\[
\forall \alpha, \beta \left[ (\alpha \text{ codes the same ordinal as } \beta) \land P(\beta) \rightarrow P(\alpha) \right]
\]

is provable in ZFC, for we may, if necessary, replace the given \( P(\alpha) \) with
the new \( \Sigma^1_2 \) formula

\[
\exists \beta \left[ (\alpha \text{ codes the same ordinal as } \beta) \land P(\beta) \right].
\]

For each countable ordinal \( \sigma \), let \( C_\sigma \) be the set of one-to-one finite
partial functions from \( \omega \) to \( \sigma \). We think of \( C_\sigma \) as a notion of forcing (see
[2]), and we write \( \Vdash_{C_\sigma} \) for the associated (weak) forcing relation. The forcing
language contains a name \( G_\sigma \) for the generic subset of \( C_\sigma \) and a name
\(\gamma_\sigma\) for the well-ordering of \(\omega\) (or a finite subset of \(\omega\)) of length \(\sigma\) induced by the bijection \(\bigcup G_\sigma\) from \(\omega\) (or a finite subset) onto \(\sigma\). Thus,

\[\emptyset \mathrel{\models} G_\sigma\]

is a generic (over the ground model \(V\)) subset of \(\check{\mathcal{C}}_\sigma\),

and \(\gamma_\sigma\) is the well-ordering of \(\check{\omega}\) (or a finite subset)

induced by \(\bigcup G_\sigma\); thus \(\gamma_\sigma\) is a code for \(\check{\sigma}\).

It is easy to check that \(C_\sigma, G_\sigma, \gamma_\sigma\) are constructible functions of \(\sigma\).

Consider a fixed countable ordinal \(\sigma\) and a code \(\alpha\) for \(\sigma\). Let \(C^*\) be a notion of forcing with respect to which every condition (weakly) forces:

\[(4)\text{ Every element of } \check{\mathcal{C}}_\sigma \text{ belongs to some generic (over } V \text{) subset of } \check{\mathcal{C}}_\sigma.\]

For example, \(C_\sigma\) itself is such a notion of forcing, but it is perhaps easier to verify (4) if we take \(C^*\) such that the power of the continuum is collapsed to \(\omega\). With respect to any such \(C^*\), every condition (weakly) forces the content of the following paragraph.

For every generic (over \(V\)) subset \(G\) of \(\check{\mathcal{C}}_\sigma\), inducing a well-ordering \(\gamma_G\) of \(\check{\omega}\) (or a finite subset) of length \(\check{\sigma}\), we have the following chain of equivalences:

\[
\begin{align*}
\check{\sigma} \in \check{A} &\iff \check{V} \models P(\check{\alpha}) & \text{by (1),} \\
&\iff P(\check{\alpha}) & \text{by Shoenfield's absoluteness theorem,} \\
&\iff P(\gamma_G) & \text{by (2) as both } \check{\alpha} \text{ and } \gamma_G \text{ code } \check{\sigma}, \\
&\iff L[G] \models P(\gamma_G) & \text{by Shoenfield again.}
\end{align*}
\]

As \(G\) is generic over \(L\) and \(\check{\gamma}_\sigma\) denotes \(\gamma_G\) in the usual interpretation of the forcing language in \(L[G]\), the last formula in our chain of equivalences is implied by \(L \models (\emptyset \mathrel{\models} P(\check{\gamma}_\sigma))\). But conversely, if in \(L\) the empty condition does not force \(P(\check{\gamma}_\sigma)\), then there is a \(p \in \check{\mathcal{C}}_\sigma\) forcing (in \(L\)) \(\neg P(\check{\gamma}_\sigma)\).

By (4), this \(p\) is in some generic \(G\), so, by the chain of equivalences, \(\check{\sigma} \notin \check{A}\). We have therefore

\[(5)\check{\sigma} \in \check{A} \iff \emptyset \mathrel{\models} (\emptyset \mathrel{\models} P(\check{\gamma}_\sigma)).\]

In the formula (5), which is forced with respect to \(C^*\), all quantifiers are restricted to \(L\). Therefore, we have (in the real world)

\[(6)\sigma \in A \iff (\emptyset \mathrel{\models} \sigma \mathrel{\models} P(\gamma_\sigma)),\]

from which it immediately follows (since \(\sigma\) was arbitrary) that \(A \in L\).
REFERENCES


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