A FORCING PROOF OF THE
KECHRIS-MOSCHOVAKIS CONSTRUCTIBILITY THEOREM

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ABSTRACT. We show, by forcing, that every subset of $\mathbb{N}$ whose
codes form a $\Sigma^1_2$ set of reals must be constructible.

In [1], Kechris and Moschovakis proved the following theorem by a game-
theoretic argument and expressed doubt whether it could be proved by the
forcing techniques of Solovay [3].

Theorem (Kechris-Moschovakis). Let $A$ be a set of countable ordinals
whose codes form a $\Sigma^1_2$ set of reals. Then $A$ is constructible.

(For details of the coding of ordinals by reals, see [1].)

The purpose of this note is to prove this theorem by forcing.

Let $A$ be as in the hypothesis of the theorem, and let $P$ be a $\Sigma^1_2$ formula
such that, whenever a real $\alpha$ codes an ordinal $\sigma$,

$$\sigma \in A \iff P(\alpha).$$

We may suppose, without loss of generality, that the statement

$$\forall \alpha, \beta \ [(\alpha \ codes \ the \ same \ ordinal \ as \ \beta) \land P(\beta) \rightarrow P(\alpha)]$$

is provable in ZFC, for we may, if necessary, replace the given $P(\alpha)$ with
the new $\Sigma^1_2$ formula

$$\exists \beta \ [(\alpha \ codes \ the \ same \ ordinal \ as \ \beta) \land P(\beta)].$$

For each countable ordinal $\sigma$, let $C_\sigma$ be the set of one-to-one finite
partial functions from $\omega$ to $\sigma$. We think of $C_\sigma$ as a notion of forcing (see
[2]), and we write $\Vdash_\sigma$ for the associated (weak) forcing relation. The forcing
language contains a name $G_\sigma$ for the generic subset of $C_\sigma$ and a name
\(\gamma_\sigma\) for the well-ordering of \(\omega\) (or a finite subset of \(\omega\)) of length \(\sigma\) induced by the bijection \(\bigcup G_\sigma\) from \(\omega\) (or a finite subset) onto \(\sigma\). Thus,

\[\emptyset \Vdash G_\sigma\] is a generic (over the ground model \(V\)) subset of \(\check{C}_\sigma\),

and \(\gamma_\sigma\) is the well-ordering of \(\check{\omega}\) (or a finite subset) induced by \(\bigcup G_\sigma\); thus \(\gamma_\sigma\) is a code for \(\check{\sigma}\).

It is easy to check that \(C_\sigma, G_\sigma\) and \(\gamma_\sigma\) are constructible functions of \(\sigma\).

Consider a fixed countable ordinal \(\sigma\) and a code \(\check{\sigma}\) for \(\sigma\). Let \(C^*\) be a notion of forcing with respect to which every condition (weakly) forces:

(4) Every element of \(\check{C}_\sigma\) belongs to some generic (over \(V\)) subset of \(\check{C}_\sigma\).

For example, \(C_\sigma\) itself is such a notion of forcing, but it is perhaps easier to verify (4) if we take \(C^*\) such that the power of the continuum is collapsed to \(\omega\). With respect to any such \(C^*\), every condition (weakly) forces the content of the following paragraph.

For every generic (over \(V\)) subset \(G\) of \(\check{C}_\sigma\), inducing a well-ordering \(\gamma_G\) of \(\check{\omega}\) (or a finite subset) of length \(\check{\sigma}\), we have the following chain of equivalences:

\[
\check{\sigma} \in \check{A} \iff \check{V} \Vdash P(\check{\sigma}) \quad \text{by (1)},
\]

\[
\longrightarrow P(\check{\sigma}) \quad \text{by Shoenfield's absoluteness theorem},
\]

\[
\longrightarrow P(\gamma_G) \quad \text{by (2) as both \(\check{\sigma}\) and \(\gamma_G\) code \(\check{\sigma}\)},
\]

\[
\longrightarrow L[G] \Vdash P(\gamma_G) \quad \text{by Shoenfield again}.
\]

As \(G\) is generic over \(L\) and \(\check{\gamma}_\sigma\) denotes \(\gamma_G\) in the usual interpretation of the forcing language in \(L[G]\), the last formula in our chain of equivalences is implied by \(L \vDash (\emptyset \Vdash \check{P}(\check{\gamma}_\sigma))\). But conversely, if in \(L\) the empty condition does not force \(P(\check{\gamma}_\sigma)\), then there is a \(p \in \check{C}_\sigma\) forcing (in \(L\)) \(\neg P(\check{\gamma}_\sigma)\).

By (4), this \(p\) is in some generic \(G\), so, by the chain of equivalences, \(\check{\sigma} \notin \check{A}\). We have therefore

(5) \[\check{\sigma} \in \check{A} \iff L \vDash (\emptyset \Vdash \check{P}(\check{\gamma}_\sigma)).\]

In the formula (5), which is forced with respect to \(C^*\), all quantifiers are restricted to \(L\). Therefore, we have (in the real world)

(6) \[\sigma \in A \iff L \vDash (\emptyset \vDash P(\gamma_\sigma)),\]

from which it immediately follows (since \(\sigma\) was arbitrary) that \(A \in L\).
REFERENCES


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