GENERALIZED FLAG MANIFOLDS
BOUND EQUIVARIANTLY

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ABSTRACT. Given a compact, connected lie group $G$ and a maximal
torus $T$, we give a simple, explicit construction of a $G$-manifold $M$ which
bounds the homogeneous space $G/T$ equivariantly.

Let $G$ be a compact, connected lie group with a maximal
torus $T$. We
will construct a compact manifold $M$ with a $G$-action, and a $G$-equivariant
imbedding $G/T \to M$ which identifies $G/T$ with the boundary of $M$. It is
known [1] that the Pontryagin and Stiefel-Whitney classes of $G/T$ vanish, so
certainly the corresponding characteristic numbers; and hence by general
results of cobordism theory, $G/T$ bounds. However, this result about the
characteristic classes requires a detailed study of the cohomology of $G/T$.
In any case, the deduction from cobordism theory does not provide any
"simple" explicit manifold bounding $G/T$, let alone equivariantly.

We start with the well-known decomposition for the lie algebra $\mathfrak{g}$ of $G$,
as an oriented $\text{ad}(T)$-module: $\mathfrak{g} = \mathfrak{t} \oplus \sum_{\alpha > 0} \mathfrak{g}_\alpha$, where $\mathfrak{t}$ is the lie algebra
of $T$ and $\mathfrak{g}_\alpha$ are irreducible, oriented $\text{ad}(T)$-planes corresponding to the
(positive) roots $\alpha : \mathfrak{t} \to \mathbb{R}$. The subspace $\mathfrak{c}_\alpha$ generated by $\mathfrak{t}$ and $\mathfrak{g}_\alpha$ is
actually a lie subalgebra [2, Chapter 6] isomorphic to $\alpha \oplus \mathfrak{su}(2)$, where
$\alpha = \text{Ker}(\alpha)$ is an abelian ideal, and $\mathfrak{su}(2)$ is generated by the coroot $H_\alpha \in \mathfrak{t}$
and $\mathfrak{g}_\alpha$. Denoting by $C_\alpha \subseteq G$ the connected subgroup corresponding to $\mathfrak{c}_\alpha$,
it is easy to see that $C_\alpha / T \cong S^2$, the two-sphere. Here is a quick proof:
$C_\alpha / T$ is a compact two-manifold, and since any compact lie group modulo
its maximal torus is simply-connected, it must be $S^2$. Note that $S^2$ acquires
a natural orientation from $\mathfrak{g}_\alpha$.

Now consider the homogeneous fibre-bundle $C_\alpha / T \to G/T \to G/C_\alpha$.
This exhibits $G/T$ as $G \times C_\alpha(S^2)$ as a $G$-space, with $G$ acting on the latter
space by left multiplication in the first factor. Our main observation is that

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1Serre [2] only discusses the semisimple case, but since $\mathfrak{g}$ is reductive, the
same argument applies.
the $C_\alpha$-action on $S^2$ is equivalent to one through $SO(3)$, hence extends to a $C_\alpha$-action on the three-disk $D^3$. This can be seen directly, if one writes out the isomorphism $C_\alpha/T \simeq S^2$ using the lie theory above. For our purposes, we can invoke "uniformization" since $C_\alpha$ preserves some complex structure [1, §12]—or equivalently, some Riemannian metric—on $S^2$. Therefore we construct the manifold $M = G \times C_\alpha(D^3)$, with $G$ acting on $M$ by left multiplication in the first factor. The obvious inclusion $G \times C_\alpha(S^2) \to G \times C_\alpha(D^3)$ then gives the required imbedding.

In a word, we have "filled in" the two-spheres in the fibre-bundle above. Note that there is a $G$-equivariant fibre-map $\pi: M \to G/C_\alpha$ with fibre $D^3$, and our homogeneous bundle above is the "boundary-bundle" of $D^3 \to M \to G/C_\alpha$.

BIBLIOGRAPHY


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