OMITTING TYPES: APPLICATION TO DESCRIPTIVE SET THEORY

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ABSTRACT. The omitting types theorem of infinitary logic is used to prove that every small $\Pi^1_1$ set of analysis or any small $\Sigma^1_1$ set of set theory is constructible.

In what follows we could use either the omitting types theorem for infinitary logic or the same theorem for what Grilliot [2] calls $(eA)$-logic. I find the latter more appealing. Suppose $\mathcal{L}$ is a finitary logical language containing the symbols of set theory as well as a constant symbol $a$ for each $a$ in the transitive set $A$. For this language we will use only $(eA)$-models, that is to say, end extensions of the model $(A,\epsilon)$. Corresponding to this restricted notion of model is a strengthened notion of proof, $(eA)$-logic. In addition to the usual finitary rules of proof, this logic contains rules $R_a$ for each $a \in A$. Rule $R_a$ says "From $\phi(b)$ for each $b \in a$, you may conclude $\forall x \in \mathcal{A}\phi(x)$." This logic satisfies both the completeness and omitting types theorems. If $A$ is admissible and $T$ is $\Sigma$ on $A$, the predicate $T_{eA}\phi$ is also $\Sigma$ on $A$. Proofs follow easily from the corresponding theorems of infinitary logic.

A $\Pi^1_1$ set is small if it has no perfect subsets. Using the theorem that every set $\Sigma^1_1$ in the parameter $a$ having a member not hyperarithmetic in $a$ has a perfect subset, a number of people have observed that every small $\Pi^1_1$ set is contained in the set $S$ defined as follows: $a \in S$ iff $a$ is hyperarithmetic in every $\beta$ with $\omega_1^\alpha \leq \omega_1^\beta$. Here $\omega_1^\alpha$ is the first ordinal not recursive in $a$. It has also been observed that $S = Q$, where $Q$ is the set of $\alpha$ which are constructible by stage $\omega_1^\alpha$ in the constructible hierarchy. Since $Q \subseteq L$, in order to prove that no small $\Pi^1_1$ set has a nonconstructible element, $^2$

Received by the editors June 15, 1973.


Key words and phrases. Constructible, perfect set, hyperarithmetic, analytic.

$^1$ These include Kechris, Sachs, and Guaspari. Guaspari claims the record of 7 different characterizations of $S$.

$^2$ This theorem was first proven in [4] and [5]. The above-quoted strengthening of the theorem to $S=Q$ has previously been proven using essentially the same forcing techniques as the original theorem.

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it suffices to show half of this equality, namely $S \subseteq Q$.

**Theorem 1.** $S \subseteq Q$.

**Proof.** Let $\alpha$ be an arbitrary set of integers and let $\sigma$ be $\omega_1^\alpha$. Consider the language for $(\omega \sigma)$-logic which contains an additional unary predicate letter $F$. Let $T$ be the ZF set theory augmented by the axioms "$F \subseteq \omega$" and each instance of "$\tau < \omega^F$, for $\tau < \sigma$. The type $D$ is $\{x \subseteq \omega \cup \{n \in x: n \in \alpha \cup \{n \notin x: n \notin \alpha\}\}$. Since $T$ has an $(\omega \sigma)$-model, either $D$ is principal in $(\omega \sigma)$-logic or $T$ has an $(\omega \sigma)$-model omitting $D$.

In the first case let $\phi(x)$ be a generator of $D$. Then $\alpha = \{n : T \vDash \epsilon \sigma \phi(x) \rightarrow \neg n \in x\}$ and $\sim \alpha = \{n : T \vDash \epsilon \sigma \phi(x) \rightarrow \neg n \in x\}$, thus $\alpha \in \Delta$ on the admissible set $L(\sigma)$ and hence is in $L(\sigma)$; thus $\alpha \in Q$.

In the second case there is an $(\omega \sigma)$-model for $T$ not containing $\alpha$. Letting $\beta$ be the interpretation of $F$ in that model, we see that $\omega_1^\alpha \subseteq \omega_1^\beta$, but $\alpha$ is not hyperarithmetic in $\beta$. Thus $\alpha \notin S$. □

**Definition.** For $x$ and $y$ hereditarily countable sets, $x$ is hyperarithmetic in $y$ if $x \in A$ for every admissible $A$ with $y \in A$. Note that for $x$ and $y$ sets of integers, this definition is equivalent to the other usual ones.

**Lemma.** If $\alpha$ is a countable ordinal and $A \subseteq \sigma$ is not hyperarithmetic in $\sigma$, then there is a well ordering of integers $<$ of type $\sigma$ with $A$ not hyperarithmetic in $<$. \[\text{Proof.}\] We use $(\sigma + 1)$ logic. Let $\ell$ be the language for that logic augmented by the binary relation symbol $<$. $T$ is ZF set theory augmented by "$<$ is a well ordering of integers of type $\sigma$". The type $D$ is $\{x \subseteq \ell \cup \{\rho \in x: \rho \in A \cup \{\rho \notin x: \rho \notin A\}\}$. As in the previous proof, if $D$ were principal, $A$ would be hyperarithmetic in $\sigma$. Thus $D$ is not principal and so $T$ has an $(\sigma + 1)$-model not containing $A$. Since the well-founded part of any model of ZF is admissible, this completes the proof. □

**Definition.** A set $A \subseteq P(\rho)$ (the power set of $\rho$) is analytic in $\sigma$ if there is a formula $\phi$ of set theory with $A$ defined by the condition: "There is a transitive set $a$ of rank $\leq \sigma$ with $x \in a$ and $(a, \epsilon) \vDash \phi(x)$.''

**Theorem 2.** If $A \subseteq P\sigma$ is analytic and has an element not hyperarithmetic in $\sigma$, then $A$ has $2^{\aleph_0}$ elements.

**Proof.** Suppose $x \in A$ is not hyperarithmetic in $\sigma$. Let $<$ be a well ordering of integers of type $\sigma$, with $x$ not hyperarithmetic in $<$. This ordering obviously induces a simple map from $\sigma$ 1-1 onto $\omega$, and hence a functional $F$ recursive in $<$ mapping $P\sigma$ 1-1 onto $P\sigma$. Clearly $F(x)$ is not
hyperarithmetic in , so it remains to show that \( \{ F(y) : y \in A \} \) is \( \Sigma^1_1 \) in .

The theorem would then follow directly from the corresponding theorem for \( \Sigma^1_1 \) sets quoted in the second paragraph. \( \{ F(y) : y \in A \} \) can be defined as the set of \( z \) satisfying "there is a binary relation \( R \) on the integers which can be mapped into \( < \) and an integer \( n \) such that \( z \) is the image under \( F \) of the transitive collapse of \( n \) and \( \langle \omega, R \rangle \models \phi(n) \)." This condition can be routinely shown to be \( \Sigma^1_1 \).

**Theorem 3.** Suppose \( A \) is \( \Sigma \) on \( HC \) (the set of hereditarily countable sets) and has a nonconstructible element, then \( A \) has \( 2^{\aleph_0} \) elements.

**Proof.** We may as well assume that \( A \) is transitive, since its transitive closure is also \( \Sigma \) on \( HC \), and \( \text{mod} \aleph_0 \) has the same cardinal as \( A \). Let \( F \) be the usual \( \Sigma \) isomorphism of \( L_\omega \) onto \( \omega_1 \). For the same reason as above, we may as well assume that for \( x \in A \), \( \{ F(y) : y \in x \} \) is also in \( A \). By these two assumptions \( A \) contains a nonconstructible set of ordinals. Let \( \sigma \) be a countable ordinal and \( x_0 \) be an element of \( (A - L) \cap \text{Po} \). Let \( \rho \geq \sigma \) be such that \( V_\rho \models \phi(x_0) \), where \( \phi \) is the \( \Sigma \) definition of \( A \). Then the set of \( y \subseteq \sigma \) satisfying "there is a transitive set \( a \) with rank \( (a) \leq \rho \) and \( y \in a \) and \( \langle a, e \rangle \models \phi(y) \)" is a subset of \( A \), analytic in \( \rho \), containing the nonconstructible element \( x_0 \). Since \( x_0 \) is not hyperarithmetic in any ordinal, this, with Theorem 2, completes the proof.

**BIBLIOGRAPHY**


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