THE DUAL OF A THEOREM OF BISHOP AND PHELPS

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ABSTRACT. We dualize a theorem of Bishop and Phelps by showing that in the dual of a Banach space the intersection of a weak* closed finite codimensional linear variety and a weak* closed convex subset $C$ contains a norm dense set of weak* support points of $C$. We use this theorem to obtain a result which is related to an abstract approximation problem of Deutsch and Morris.

If $C$ is a weak* closed convex subset of $E^*$, the dual of a Banach space $E$, then by a weak* support point of $C$ we mean a point $z^* \in C$ for which there exists $z \in E \setminus \{0\}$ such that $S_C(z) = (z, z^*)$. ($S_C$ is the support function for $C$ and is defined for each $x \in E$ by $S_C(x) = \sup \{ \langle x, x^* \rangle | x^* \in C \}$.)

In [5], Phelps showed that the set of weak* support points of $C$ is large in the sense of the following theorem.

**Theorem 1 [Phelps].** If $E$ is a Banach space and $C$ is a weak* closed convex subset of $E^*$, then the weak* support points of $C$ are norm dense in the norm boundary of $C$.

As a consequence of Theorem 1 and the following lemma of Bishop-Phelps [1, Lemma 4], we will obtain a dual to [1, Theorem 4]. We remark that although [1, Lemma 4] is stated only for Banach spaces, its proof is valid in any topological vector space.

**Lemma 2 [Bishop-Phelps].** Suppose $M$ is a closed subspace of finite codimension in a topological vector space $X$, and that $C$ is a convex subset of $X$. Suppose $x_0$ is a support point of $C \cap M$ in the subspace $M$. Then $x_0$ is a support point of $C$.

By the polar $C^o$ of a set $C \subseteq E$, we mean the set $\{ x^* \in E^* | S_C(x^*) \leq 1 \}$. If $N$ is a subspace of $E$, then $N^\perp$ denotes the annihilator of $N$ in $E^*$, i.e. $N^\perp = \{ n^* \in E^* | \langle n, n^* \rangle = 0 \text{ for each } n \in N \}$.

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Theorem 3. Let $C$ be a closed convex subset of the Banach space $E$, and let $N$ be a finite-dimensional subspace of $E$. Suppose $z^* \in N^\perp \cap \text{bdry } C^\circ$. Then for each $\epsilon > 0$ there exists a weak* support point $w^*$ of $C^\circ$ such that $w^* \in N^\perp$ and $\|w^* - z^*\| \leq \epsilon$.

Proof. We identify the Banach spaces $E/N$ and $N^\perp$. Recall that $N^\perp$ with the relative $\sigma(E^\ast, E)$ topology is topologically isomorphic with $(E/N)^\ast$ with the $\sigma(E^\ast, E/N)$ topology. Thus the set $C^\circ \cap N^\perp$ is a $\sigma(N^\perp, E/N)$ closed convex subset of $N^\perp$. If $z^* \in \text{norm bdry } (C^\circ \cap N^\perp)$ in $N^\perp$, then by Theorem 1 applied to $N^\perp$, there exists a $\sigma(N^\perp, E/N)$ support point $w^*$ of $C^\circ \cap N^\perp$ in $N^\perp$ such that $\|w^* - z^*\| \leq \epsilon$. By Lemma 2 the element $w^*$ is a $\sigma(E^\ast, E)$ support point of $C^\circ$ in $E^\ast$.

If $z^* \notin \text{norm bdry } (C^\circ \cap N^\perp)$ in $N^\perp$, then we show that $z^*$ is itself a weak* support point of $C^\circ$. Since $z^* \in \text{bdry } C^\circ$, there exists an element $y^* \in E^\ast \setminus C^\circ$ such that the segment $[z^*, y^*] \subset E^\ast / C^\circ$. (Otherwise, $z^*$ is an element of the core of $C^\circ$, and since $C^\circ$ is closed and $E^\ast$ is of the second category in itself, the core of $C^\circ$ is equal to the interior of $C^\circ$.) Let $M = \text{span } (N^\perp \cup \{y^\ast\})$ and note that $N^\perp$ is a hyperplane in $M$; if we show that $N^\perp$ supports $C^\circ \cap M$ at $z^\ast$, then from Lemma 2, we can conclude that the point $z^\ast$ is a weak* support point of $C^\circ$. It suffices to show that $C^\circ$ is disjoint from the open half space $\{n^\ast + ry^* | n^\ast \in N^\perp \text{ and } r > 0\}$ in $M$ defined by $N^\perp$. If $n^\ast + ry^* \in C^\circ$ where $r > 0$, then (since $z^\ast$ is in the $N^\perp$-interior of $(C^\circ \cap N^\perp)$) there exists $w^\ast \in C^\circ$ and $\lambda \in (0, 1)$ such that $z^\ast = \lambda n^\ast + (1 - \lambda)w^\ast$. Thus the triangle with vertices $n^\ast + ry^*$, $n^\ast$, and $w^\ast$ is in $C^\circ$, and this clearly contradicts the fact that $[z^*, y^*] \subset E^\ast / C^\circ$. This completes the proof.

Let $C$ be a closed convex subset of the Banach space $E$, and denote by $P(C)$ the set of support functionals of $C$.

We are going to prove that the intersection of a weak* closed flat of finite codimension with the polar of a closed convex bounded subset $C$ of a Banach space contains a norm dense set of support functionals of $C$. This will then be shown to be related to an abstract approximation problem of Deutsch and Morris [2].

Proposition 4. Let $E$ be a Banach space and $N$ a finite-dimensional subspace of $E$. Suppose $C \subset E$ is closed convex and bounded and $0 \in C$. If $x^\ast \in E^\ast$ and $S_C(x^\ast) = 1$, then for each $\epsilon > 0$ there exists $z^\ast \in P(C) \cap B(x^\ast; \epsilon) \cap (x^\ast + N^\perp)$ such that $S_C(z^\ast) = 1$.

Proof. Since $S_C(x^\ast) = 1$, we have $x^\ast \in \text{bdry } C^\circ$, hence $0 \in (\text{bdry } (C^\circ - x^\ast)) \cap N^\perp$. According to Theorem 3 there exists a weak* support point $w^\ast$ of $C - x^\ast$ such that $\|w^\ast\| \leq \epsilon$ and $w^\ast \in N^\perp$. Let $z^\ast = w^\ast + x^\ast$; it
only remains to show that $\mathbf{z}^* \in \mathbb{P}(C)$ and $S_C(z^*) = 1$. Since $z^*$ is clearly a weak* support point of $C^\circ$, there exists $z \in \mathbb{E}\setminus\{0\}$ satisfying $S_C(z) = \langle z, z^* \rangle$. Because $z \neq 0$ there exists $n^* \in \mathbb{E}^*$ such that $(z, n^*) > 0$. Since $C$ is bounded, we know $C^\circ$ is radial at 0; hence there exists $\lambda > 0$ such that $\lambda n^* \in C^\circ$ so

$$0 < \langle z, \lambda n^* \rangle \leq S_C(z).$$

Without loss of generality we can suppose

$$S_C(z) = \langle z, z^* \rangle = 1.$$

Thus $z \in C$ (by the bipolar theorem) and we have

$$S_C(z^*) = \langle z, z^* \rangle = 1 \text{ since } z^* \in C^\circ.$$

This completes the proof.

The preceding proposition is related to an abstract approximation problem of Deutsch and Morris [2] called "property (SAIN)" for "simultaneous approximation and interpolation which is norm preserving." In the present context this property (which we call "property (S)") is the following:

If $E$ is a Banach space, $M$ is a dense subset of $E^*$, and $N$ is a finite-dimensional subspace of $E^{**}$, then the triple $(E^*, M, N)$ has property (S) if for each $\epsilon > 0$ and $x^* \in E^*$, there exists $z^* \in M$ satisfying

$$\|z^* - x^*\| < \epsilon, \quad \|z^*\| = \|x^*\|, \text{ and } z^* - x^* \in N^\perp.$$

Deutsch and Morris established in [2, Theorem 2.3] that in case $M$ is a linear subspace of $E^*$, then $(E^*, M, N)$ has property (S) only if each element of $N$ either attains its norm at points of $M$ or not at all. This raises the question of what happens if $M$ is the norm dense subset $\mathbb{P}(B)$ of $E^*$ ($B$ is the unit ball of $E$)? Since $\mathbb{P}(B)$ is not in general convex (a standing hypothesis on the set $M$ in previous theorems about property (S)), the techniques of [2] do not apply. However, as a corollary to Proposition 4, we obtain the following answer to the question raised above.

**Corollary 5.** Let $E$ be a Banach space and $B$ the unit ball of $E$. The triple $(E^*, \mathbb{P}(B), N)$ has property (S) for each finite-dimensional subspace $N \subseteq E$.

We remark that in [4] Lambert showed that in the case where $E = c_0$ and $B$ is the unit ball of $c_0$, that the triple $(l_1, \mathbb{P}(B), N)$ has property (S) for each finite-dimensional subspace $N$ of $l_\infty$. That this result does not hold for general Banach spaces $E$ and finite-dimensional subspaces $N$ of $E^{**}$ is shown by the following example.
Example. There exists a Banach space $E$ and a one-dimensional subspace $N$ of $E^{**}$ such that the triple $(E^*, P(B), N)$ does not have property (S).

Let $E = c_0$ with an equivalent norm such that $E^* = l_1$ is strictly convex; let $x^* \in S(l_1) \setminus P(c_0)$, and choose $x^{**} \in S(E^{**})$ so that $\langle x^{**}, x^* \rangle = 1$, and let $N = Rx^{**}$. Then

$$\left( x^{**} + (x^{**})^{-1}(0) \right) \cap B^* = \{ x^* \}$$

since $S(l_1)$ is strictly convex; thus

$$S(E^*) \cap P(B) \cap (x^* + (x^{**})^{-1}(0)) = \emptyset.$$ 

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