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CHARACTERS AND GENEROSITY OF PERMUTATION GROUPS

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ABSTRACT. A necessary and sufficient character condition is obtained for a group G to be generously k-fold transitive. This is similar to an old theorem of Frobenius on multiply transitive groups.

Let G be a permutation group on a set Ω of n points. Let χ^λ be the irreducible character of the symmetric group S_n corresponding to the partition λ of n. Let k, r and s be nonnegative integers such that $k = r + s$. It is a well-known fact, due essentially to Frobenius, that G is k-fold transitive on Ω if and only if $(\chi^\lambda, \chi^\mu)_G = \delta_{\lambda\mu}$ for any characters χ^λ, χ^μ of S_n with dimension $\chi^\lambda \leq r$, dimension $\chi^\mu \leq s$. Here $\delta_{\lambda\mu}$ is equal to 1 if $\lambda = \mu$ and equals 0 otherwise; and the dimension of the character χ^λ corresponding to the partition $\lambda = (\lambda_1, \cdots, \lambda_i)$ is defined to be $n - \lambda_1$ (see Lemma 2 (b) for an alternative definition). For a proof of this result see [5, 14.3].

Following P. M. Neumann we say that G is generously k-fold transitive if for any $\Delta \subseteq \Omega$ with $|\Delta| = k + 1$, the setwise stabilizer G_Δ acts as the symmetric group S_{k+1} on Δ. The aim of this note is to give a necessary and sufficient condition similar to the one above for a k-fold transitive group to be generously k-fold transitive.

Theorem. *Let G be k-fold transitive on a set Ω of n points, $n > 2k$. Let r, s be integers such that $r \geq 0$, $s \geq 2$ and $r + s = k + 1$. Then G is generously k-fold transitive if and only if $(\chi^\lambda, \chi^\mu)_G = \delta_{\lambda\mu}$ for any characters χ^λ, χ^μ of S_n such that* dimension $\chi^\lambda = r$, dimension $\chi^\mu = s$ *and* $\chi^\mu \neq \chi^{(n-s,s)}$.

A special case of this result was also obtained by E. Bannai [1].

Notation. Let $r \leq n$; we write $\Omega^{(r)}$, $\Omega^{\{r\}}$ for the S_n-space of ordered, unordered r-point subsets of Ω, respectively. We denote by $\pi^{(r)}$, $\pi^{\{r\}}$ the corresponding permutation characters. For the definitions and some properties of the irreducible characters of S_n see [2], [4] or [8].

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If H is a permutation group on a set Γ, we write $\operatorname{orb}(H, \Gamma)$ for the set of orbits of H on Γ.

The next two lemmas are well known.

Lemma 1. *Let G act on the sets Γ_1, Γ_2, and let π_1 and π_2 be the corresponding permutation characters. Then*

$$(\pi_1, \pi_2)_G = |\operatorname{orb}(G, \Gamma_1 \times \Gamma_2)|.$$

If G is transitive on Γ_1 then $(\pi_1, \pi_2)_G = |\operatorname{orb}(G_\alpha, \Gamma_2)|$ for any point α of Γ_1.

Lemma 2. *Let $r \leq n$. Then*

(a) $\pi^{\{r\}} = \Sigma_{i=0}^{r} \chi^{(n-i,i)}$;

(b) $\pi^{(r)} = \Sigma_\lambda a_\lambda \chi^\lambda$ *with* $a_\lambda > 0$, *where the summation runs over all partitions λ of n of dimension less or equal to r*;

(c) $(\pi^{(r)}, \chi^{(n-r,r)})_{S_n} = 1$.

Proof. (a) See, eg., [8, 4.2].

(b) [4].

(c) This follows easily from (a) and Lemma 1.

Proof of the Theorem. The group G is k-fold transitive, and so by the Theorem [5, 14.3] mentioned at the beginning, $(\chi^\lambda, \chi^\mu)_G = \delta_{\lambda\mu}$ for characters χ^λ, χ^μ of S_n with $\dim \chi^\lambda + \dim \chi^\mu \leq k$. Hence the character condition of the Theorem is by Lemma 2 equivalent to

$$(\pi^{(r)}, \pi^{(s)} - \pi^{\{s\}})_G = (\pi^{(r)}, \pi^{(s)} - \pi^{\{s\}})_{S_n}. \tag{1}$$

Now by Lemma 1, (1) is equivalent to

$$\begin{aligned} &|\operatorname{orb}(S_n, \Omega^{(r)} \times \Omega^{(s)})| - |\operatorname{orb}(G, \Omega^{(r)} \times \Omega^{(s)})| \\ &\qquad = |\operatorname{orb}(S_n, \Omega^{(r)} \times \Omega^{\{s\}})| - |\operatorname{orb}(G, \Omega^{(r)} \times \Omega^{\{s\}})|. \end{aligned} \tag{2}$$

Let $((\alpha_1, \cdots, \alpha_r), (\beta_1, \cdots, \beta_s)) \in \Omega^{(r)} \times \Omega^{(s)}$ and assume that $\{\alpha_1, \cdots, \alpha_r\} \cap \{\beta_1, \cdots, \beta_s\} \neq \emptyset$. Then $|\{\alpha_1, \cdots \alpha_r, \beta_1, \cdots, \beta_s\}| \leq k$. But G is k-fold transitive, and so

$$((\alpha_1, \cdots, \alpha_r), (\beta_1, \cdots, \beta_s))^G = ((\alpha_1, \cdots, \alpha_r), (\beta_1, \cdots, \beta_s))^{S_n}.$$

Hence

$$|\mathrm{orb}(S_n, \Omega^{(r)} \times \Omega^{(s)})| - |\mathrm{orb}(G, \Omega^{(r)} \times \Omega^{(s)})|$$
$$= |\mathrm{orb}(S_n, \Omega^{(r)} \dot{\times} \Omega^{(s)})| - |\mathrm{orb}(G, \Omega^{(r)} \dot{\times} \Omega^{(s)})|,$$

where

$$\Omega^{(r)} \dot{\times} \Omega^{(s)} = \{((\alpha_1, \cdots, \alpha_r), (\beta_1, \cdots, \beta_s)) \in \Omega^{(r)} \times \Omega^{(s)} \mid \{\alpha_1, \cdots, \alpha_r\} \cap \{\beta_1, \cdots, \beta_s\} = \emptyset\}.$$

We obtain a similar expression also for the right-hand side of (2). Hence (2) is equivalent to

$$(3)\qquad |\mathrm{orb}(G, \Omega^{(r)} \dot{\times} \Omega^{(s)})| - |\mathrm{orb}(G, \Omega^{(r)} \dot{\times} \Omega^{\{s\}}| = |\mathrm{orb}(S_n, \Omega^{(r)} \dot{\times} \Omega^{(s)})| - |\mathrm{orb}(S_n, \Omega^{(r)} \dot{\times} \Omega^{\{s\}})|.$$

But S_n is $(k+1)$-fold transitive, and so

$$|\mathrm{orb}(S_n, \Omega^{(r)} \dot{\times} \Omega^{(s)})| = 1 = |\mathrm{orb}(S_n, \Omega^{(r)} \dot{\times} \Omega^{\{s\}})|.$$

Thus (3) holds if and only if

$$(4)\qquad |\mathrm{orb}(G, \Omega^{(r)} \dot{\times} \Omega^{(s)})| = |\mathrm{orb}(G, \Omega^{(r)} \dot{\times} \Omega^{\{s\}})|,$$

and (4) is equivalent to

$$(5)\qquad |\mathrm{orb}(G_{\alpha_1 \cdots \alpha_r}, (\Omega \backslash \{\alpha_1, \cdots, \alpha_r\})^{(s)})| = |\mathrm{orb}(G_{\alpha_1 \cdots \alpha_r}, (\Omega \backslash \{\alpha_1, \cdots, \alpha_r\})^{\{s\}})|$$

for any $\alpha_1, \cdots, \alpha_r \in \Omega$.

Now (5) is equivalent by [6, 3.3] to saying that $G_{\alpha_1 \cdots \alpha_r}$ is generously s-fold transitive on $\Omega \backslash \{\alpha_1, \cdots, \alpha_r\}$ for any $\alpha_1, \cdots, \alpha_r \in \Omega$, which in turn is equivalent to G being generously $(r+s)$-fold transitive on Ω by [6, 3.1]. The proof is now complete.

Remark 1. In the last section of [3] Frobenius obtained the character table of M_{24}. He remarks that the characters $\chi^{(21,1)}$ and $\chi^{(21,2,1)}$ of S_{24} remain irreducible when restricted to M_{24}. The Theorem explains this remarkable fact, since M_{24} is generously 5-fold transitive [6, 6.3].

Remark 2. A similar theorem for almost generous transitivity (as defined in [6]) can be found in [7].

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REFERENCES

1. E. Bannai, *A note on characters of normal subgroup of multiply transitive permutation groups*, J. Fac. Sci. Univ. Tokyo Sect. IA **20** (1973), 373–376.

2. A. J. Coleman, *Induced representations with applications to S_n and $GL(n)$*, Queen's Papers in Pure and Appl. Math., no. 4, Queen's University, Kingston, Ont., 1966. MR 34 #2718.

3. G. Frobenius, *Über die Charactere der mehrfach transitiven Gruppen*, Berliner Berichte, 1904, 558–571.

4. R. A. Liebler and M. R. Vitale, *Ordering the partition characters of the symmetric group*, J. Algebra **25** (1973), 487–489.

5. H. Nagao, *Multiply transitive permutation groups*, Lecture Notes, California Institute of Technology, Pasadena, 1967.

6. P. M. Neumann, *Generosity and characters of multiply transitive permutation groups* (submitted for publication).

7. J. Saxl, *Multiply transitive permutation groups*, Ph.D. Thesis, Oxford, 1973.

8. E. Snapper, *Group characters and nonnegative integral matrices*, J. Algebra **19** (1971), 520–535. MR 44 #1748.

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