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AN EXTENSION OF RUNGE'S THEOREM

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ABSTRACT. Every uniformly continuously differentiable function on the compact set X in the complex plane is the uniform limit of rational functions with poles off X.

The usual hypotheses of Runge's theorem demand that the function f, to be approximated, have an analytic extension to a neighborhood of the set X on which the approximation is to take place. Our result only places conditions on f on X itself, and implies the usual form of Runge's theorem.

Definition. On a compact set X in the complex plane $\mathbf{C}$, a continuous complex-valued function f is said to be continuously uniformly differentiable (written $f \in D^{UC}(X)$) if there is a continuous function f' on X s.t. for every $\epsilon > 0$ there exists a $\delta > 0$ s.t. $\forall z \in X$ and $\forall w, w' \in X$, $w \neq w'$, with $|w-z| < \delta$, $|w'-z| < \delta$, we have

$$\left| \frac{f(w') - f(w)}{w' - w} - f'(z) \right| < \epsilon. \tag{1}$$

Let $R(X)$ denote the uniform closure on X of the rational functions with poles off X.

Theorem. $D^{UC}(X) \subseteq R(X)$.

Proof. We first observe that from (1), it follows by the Whitney extension theorem [W, p. 65] that f has a C^1 extension (still called f) to all of $\mathbf{C}$. Next, it follows from (1) by an elementary argument that if z_0 is a point of X for which there exist two sequences $\{z_n\}$ and $\{z_n'\}$ of points of X that converge to z_0 along two different lines through z_0, then the Cauchy-Riemann conditions hold at z_0. Now we remark that z_0 satisfies the above condition if it is a point of density of X with respect to planar Lebesgue measure $dx\,dy$. For if we let $E_n = \{\theta : z_0 \pm re^{i\theta} \notin X,\ 0 < r < 2^{-n}\}$, then all E_n must have measure zero at a point of density z_0. But almost every point of

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X is a point of density of X. Thus, we have proved that if $f \in D^{UC}(X)$ it has a C^1 extension to $\mathbf{C}$ that satisfies the Cauchy-Riemann equations at almost all points of X. Finally, by a slight variation of the proof of [B, Corollary 3.22, p. 160], we conclude that $f \in R(X)$. Indeed, if we take $\mu \perp R(X)$ and let $\mu^\wedge$ be its Cauchy transform we have

$$\int f\,d\mu = -\frac{1}{\pi} \iint_X \frac{\partial f}{\partial \bar{z}} \mu^\wedge(z)\,dx\,dy.$$

But $\partial f/\partial \bar{z} = 0$. a.e. on X and $\mu^\wedge(z) = 0$ off X so that $\mu \perp f$ and, hence, $f \in R(X)$ by the Hahn-Banach theorem.

REFERENCES

[B] Andrew Browder, *Introduction to function algebras*, Benjamin, New York, 1969. MR 39 #7431.

[W] Hassler Whitney, *Analytic extensions of differentiable functions defined in closed sets*, Trans. Amer. Math. Soc. 36 (1934), 63–89.

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