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EXISTENCE AND REPRESENTATION OF SOLUTIONS OF PARABOLIC EQUATIONS[1]

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ABSTRACT. Let L be a linear, second order parabolic operator in divergence form and let Q be a bounded cylindrical domain in E^{n+1}. Let $\partial_p Q$ denote the parabolic boundary of Q. To each continuous function f on $\partial_p Q$ there is a unique solution u of the boundary value problem $Lu = 0$ in Q, $u = f$ on $\partial_p Q$. Moreover, for the given L and Q, to each $(x, t) \in Q$ there is a unique nonnegative measure $\mu_{(x,t)}$ with support on $\partial_p Q$ such that the solution of the boundary value problem is given by $u(x, t) = \int_{\partial_p Q} f \, d\mu_{(x,t)}$.

I. Introduction and preliminary results. Let $\Omega \subset E^n$ be a bounded domain with compact boundary, $\partial\Omega$, and let $T > 0$. Set $Q = \Omega \times (0, T]$ and let $\partial_p Q = \{\partial\Omega \times [0, T]\} \cup \{\Omega \times (0)\}$ denote the parabolic boundary of Q. Write $u,_i = \partial u/\partial x_i$ and $u_t = \partial u/\partial t$.

The given functions and solutions will lie in multidimensional L^p spaces and the Sobolev space $L^2[0, T; H^{1,2}(\Omega)]$. These spaces are defined in detail by Aronson and Serrin [2]. The parabolic operator under consideration is defined by

$$Lu = u_t - \{a_{ij}(x, t)u,_i + d_j(x, t)u\},_j - b_j(x, t)u,_j - c(x, t)u \tag{1}$$

where products involving repeated indices i or j are summed for $1 \leq i, j \leq n$. The results obtained are as follows:

Theorem 1. *Let $f \in C(\partial_p Q)$. There is a unique weak solution u of the boundary value problem $Lu = 0$ in Q, $u = f$ on $\partial_p Q$.*

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Theorem 2. *Let L and Q be given. To each $(x, t) \in Q$ there is a unique nonnegative measure $\mu_{(x,t)}$ on $\partial_p Q$ such that the solution u corresponding to the data f found in Theorem 1 is given by $u(x, t) = \int f \, d\mu_{(x,t)}$.*

Theorem 1 is an extension of an existence result obtained by Aronson [1] restated below as Theorem B.

The coefficients appearing in the operator L will be assumed to satisfy the following assumptions collectively called (H):

H.1. The $a_{ij}(x, t)$ are measurable functions in (x, t) with

(a) $|a_{ij}(x, t)| \leq M < \infty$ almost everywhere in Q, and

(b) for some $\lambda > 0$, $a_{ij}(x, t) z_i z_j \geq \lambda |z|^2 = \lambda \sum_{i=1}^n z_i^2$ for all $z \in E^n$ and almost all $(x, t) \in Q$.

H.2. $c(x, t) \in L^q[0, T; L^p(\Omega)]$ for some pair p, q satisfying

$$(*) \qquad 1 < p, q \leq \infty, \qquad n/2p + 1/q < 1.$$

H.3. $b_j(x, t), d_j(x, t) \in L^q[0, T; L^p(\Omega)]$ for some pair p, q satisfying

$$(**) \qquad 2 < p, q \leq \infty, \qquad n/2p + 1/q < 1/2.$$

For easy reference one basic definition and three basic theorems are stated here without proof.

Definition 1. Let L be as described as above. Assume $G(x, t) \in L^q[0, T; L^p(\Omega)]$ where p, q satisfy $(*)$ and $F_i(x, t) \in L^q[0, T; L^p(\Omega)]$ where p, q satisfy $(**)$. $u(x, t)$ is called a *weak solution of the boundary value problem*

$$(2) \qquad Lu = G(x, t) + \{F_i(x, t)\},_i \quad \text{in } Q,$$

$$(3) \qquad \begin{aligned} u(x, t) &= 0 \qquad \text{on } \mathcal{S} = \Omega \times [0, T], \\ u(x, t) &= u_0(x) \quad \text{on } \Omega \end{aligned}$$

if

(a) $u \in L^2[\delta, T; H^{1,2}_{\text{loc}}(\Omega)] \cap L^\infty[\delta, T; L^2_{\text{loc}}(\Omega)]$ for each $\delta > 0$,

(b) $u_0(x) \in L^2(\Omega)$,

and if, for each $v(x, t) \in C^1(Q) \cap C^0(\overline{Q})$ which vanishes in a neighborhood of $\mathcal{S}$,

$$(c) \qquad \begin{aligned} &\int_0^T \int_\Omega [a_{ij} u,_i v,_j + d_j v,_j u - b_j u,_j v - cuv - uv_t] \, dx \, dt \\ &= \int_0^T \int_\Omega [Gv - F_i v,_i] \, dx \, dt + \int_\Omega u_0(x) v(x, 0) \, dx, \end{aligned}$$

and

(d) $\lim_{t\downarrow 0} \int_\Omega u(x, t)v(x, t)\,dx = \int_\Omega u_0(x)v(x, 0)\,dx.$

Aronson and Serrin [2] have shown that every weak solution of (2) in Q has a representative that is continuous in Q. Henceforth, u will denote the continuous representative of a given weak solution.

Theorem A (*Maximum Principle*). *Suppose L satisfies* (H) *and let u be the weak solution of $Lu = 0$ in Q. If $u \in C^0(\overline{Q})$ and $m_1 \le u \le m_2$ on $\partial_p Q$, then*

$$\min(m_1, 0) - \mathcal{C}k_1 \le u(x, t) \le \max(m_2, 0) + \mathcal{C}k_2 \quad \text{in } \overline{Q}$$

where $\mathcal{C}$ depends on Q and the data in (H) *and*

$$k_i = |m_i| \left(\sum_{j=1}^{n} \|d_j\|_{p,q} + \|c\|_{p,q} \right) \quad \text{for } i = 1, 2.$$

A proof of this theorem can be found in [2].

Theorem B (*Existence*). *Suppose L satisfies* (H) *and $u_0(x)$, $F_i(x, t)$, and $G(x, t)$ are as described in Definition* 1. *Then there is a unique weak solution u of the boundary value problem* (2), (3). *Moreover, if $\partial\Omega$ is smooth and $u_0(x) \in C_0^0(\Omega)$, then $u \in C(\overline{Q})$.*

A proof of this theorem can be found in [1].

Theorem C (*Energy Inequality*). *Let u be a solution of $Lu = 0$ in Q with initial values $u_0 \in L^2(\Omega)$ and let $\zeta = \zeta(x)$ be a nonnegative smooth function such that $\zeta u \in L^2[0, T; H_0^{1,2}(\Omega)]$. Then there is a positive constant $\mathcal{C}$ such that*

$$\|\zeta u\|_{2,\infty}^2 + \|\zeta u_x\|_{2,2}^2 \le \mathcal{C}\left\{\int_\Omega \zeta^2 u_0^2\,dx + \|\zeta_x u\|_{2,2}^2\right\}.$$

A proof of this theorem can be found in [2]. Finally, weak solutions of $Lu = 0$ in Q are locally Hölder continuous with exponent depending on the distance of the points to $\partial_p Q$.

II. Existence theorem.

Theorem 1. *Let L and Q be as described above. Let $f(x, t)$ be continuous on $\mathcal{S}$ and satisfy $f(x, 0) \in L^2(\Omega)$. Then there is a unique weak solution u of the boundary value problem*

$$(4) \qquad \begin{aligned} Lu &= 0 && \textit{in } Q, \\ u(x, t) &= f(x, t) && \textit{on } \partial_p Q. \end{aligned}$$

Proof. Note that f is continuous on $\mathcal{S}$, a compact set; hence f can be continuously extended to $\overline{Q}$. Let $F(x, t)$ denote this extension. Theorem B can be used to solve the boundary value problem $Lu = 0$ in Q, $u(x, t) = 0$ on $\mathcal{S}$, $u(x, 0) = f(x, 0) - F(x, 0)$ on Ω. Thus, the theorem will follow if the boundary value problem $Lu = 0$ in Q, $u(x, t) = F(x, t)$ on $\partial_p Q$ can be solved.

For the present assume $\partial\Omega$ is smooth. Approximate F on $\partial_p Q$ by polynomials $p^k(x, t)$ in the supremum norm so that on $\partial_p Q$

$$m_1 = \min_{\partial_p Q} F \le p^k(x, t) \le \max_{\partial_p Q} F = m_2.$$

Extend the domain of p^k to $\overline{Q}$ so that the extension $P^k(x, t) \in C^2(Q)$. Theorem B can be applied to solve the boundary value problem $Lv^k = -LP^k$ in Q, $v^k = 0$ on $\partial_p Q$.

Define $u^k(x, t) = v^k(x, t) + P^k(x, t)$. Then u^k satisfies

$$(5) \qquad \begin{aligned} Lu^k &= 0 && \text{in } Q, \\ u^k(x, t) &= p^k(x, t) && \text{on } \partial_p Q. \end{aligned}$$

The remainder of the proof consists of showing

(A) The solution u^k is independent of the extension P^k of p^k to $\overline{Q}$.

(B) A subsequence of the u^k can be obtained which converges weakly in $L^2[\delta, T; H^{1,2}_{\mathrm{loc}}(\Omega)]$ for each $\delta > 0$ to a weak solution of $Lu = 0$ in Q.

(C) A subsequence of that obtained in (B) converges uniformly on all compact subsets of Q.

(D) The smoothness assumption on $\partial\Omega$ is removed.

(A) Let P^k and $\overline{P}^k$ be two extensions of p^k to $\overline{Q}$ with $P^k, \overline{P}^k \in C^2(Q)$ and let $u^k, \overline{u}^k$ denote the corresponding solutions to (5). Then, since $[P^k(x, t) - \overline{P}^k(x, t)] \in L^2[0, T; H^{1,2}_0(\Omega)]$ and $\lim_{t \downarrow 0} [P^k(x, t) - \overline{P}^k(x, t)] = 0$, it follows that $U^k(x, t) \equiv u^k(x, t) - \overline{u}^k(x, t)$ satisfies $LU^k = 0$ in Q, $U^k = 0$ on $\partial_p Q$. Hence, by Theorem B, $U^k \equiv 0$ on Q. Therefore, $P^k(x, t)$ may be assumed to be a polynomial.

(B) Since $\partial\Omega$ is smooth and $Lu^k = 0$ in Q, $u^k \in C(\overline{Q})$ and, by Theorem A,

$$\overline{m}_1 = \min(m, 0) + \mathcal{C}k_1 \le u^k(x, t) \le \max(m_2, 0) + \mathcal{C}k_2 = \overline{m}_2$$

on $\overline{Q}$. Define

$$|||g|||_Q \equiv \sup_{\delta>0} \{\delta[\|g_x\|^2_{2,2,Q'} + \|g\|^2_{2,\infty,Q'}]^{1/2}\} + \sup_Q |g|$$

where $Q' = \{x \in \Omega;\ \mathrm{dist}(x, \partial\Omega) > \delta\} \times (\delta, T]$. Set $m = \max(m_2, -m_1) \geq 0$. Then, by Theorem C, $|||u^k|||_Q \leq \mathcal{C}m$. Hence, on each compact subcylinder C of Q

$$\|u^k_x\|^2_{2,2,C} + \|u^k\|^2_{2,\infty,C} \leq [\mathcal{C}m/\mathrm{dist}(C, \mathcal{C}Q)]^2. \tag{6}$$

Let $\{C^j\}$ be a sequence of open cylinders with $\overline{C}^j \subset C^{j+1}$ and $C^j \uparrow Q$. On C^1, the weak compactness of $L^2[H^{1,2}(C)]$ and (6) imply there is a subsequence $\{u^{1,k}\}$ of $\{u^k\}$ which converges weakly in $L^2[H^{1,2}(C^1)]$ to u. Having obtained the sequence $\{u^{j,k}\}$ for C^j, the weak compactness of $L^2[H^{1,2}(C^{j+1})]$ and (6) imply there is a subsequence $\{u^{j+1,k}\}$ of $\{u^{j,k}\}$ which converges weakly in $L^2[H^{1,2}(C^{j+1})]$. Since $\{u^{j+1,k}\} \subset \{u^{j,k}\}$, all of the sequences $\{u^{j,k}\}$ converge weakly to u in $L^2[H^{1,2}(C)]$ for any compact subcylinder C of Q. Set $u^j = u^{j,j}$. Then u^j converges weakly to u in $L^2[H^{1,2}(C)]$. Hence, u satisfies $Lu = 0$ weakly in Q and, by Theorem C, $|||u|||_Q \leq \mathcal{C}m$.

(C) Since the u^j satisfy $Lu = 0$ in Q, they are Hölder continuous on any cylinder C with $\overline{C} \subset Q$. Hence, on each such cylinder, the family $\{u^j\}$ is equicontinuous. Then, by Arzela's theorem, there is a subsequence of $\{u^j\}$ which converges uniformly on C. By using the sequence $\{C^j\}$ given in (B) and the diagonalization process again, a subsequence of $\{u^j\}$ is obtained which converges uniformly on any compact subset of Q to u. It follows from the uniform convergence of p^k to F on $\partial_p Q$ that u is the weak solution of the boundary value problem. The uniqueness of u follows from Theorem B.

(D) Suppose $\partial\Omega$ is not smooth. Then approximate Ω by smooth domains Ω^k with $\overline{\Omega}^k \subset \Omega^{k+1}$, $\Omega^k \uparrow \Omega$, and the argument in (C) applies to each cylinder $Q^k = \Omega^k \times (0, T]$. Then the discussion in parts (B) and (C) can be repeated to give the unique weak solution u in Q.

II. Representation theorem. In this section the following representation is obtained.

Theorem 2. *Let L and Q be as described above. Then, for each $(x, t) \in Q$, there is a unique nonnegative measure $\mu_{(x,t)}$ concentrated on $\partial_p Q$ such that, for each continuous function f on $\partial_p Q$, the solution u of the boundary value problem* (4) *is given by*

$$u(x, t) = \int_{\partial_p Q} f\, d\mu_{(x,t)}.$$

Moreover, for constants a, A *such that* $0 < a \leq \int_{\partial_p Q} d\mu_{(x,t)} \leq A$, *it follows that the solution* u *satisfies*

$$\min_{\partial_p Q} (af(x, t), Af(x, t)) \leq u(x, t) \leq \max_{\partial_p Q} (af(x, t), Af(x, t)).$$

Proof. Define for each $(x, t) \in \overline{Q}$ the functional $\Lambda_{(x,t)}$ on $C(\partial_p Q)$ by

$$\begin{aligned} \Lambda_{(x,t)} f &= u(x, t) \quad \text{on } Q, \\ &= f(x, t) \quad \text{on } \partial_p Q. \end{aligned}$$

$\Lambda_{(x,t)}$ is clearly a positive linear functional and the desired result follows from the Riesz representation theorem.

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