ON THE SCALAR CURVATURE AND SECTIONAL CURVATURES OF A TOTALLY REAL SUBMANIFOLD

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ABSTRACT. For a totally real minimal submanifold of a complex space form, pinching for scalar curvature implies pinching for sectional curvatures.

Let $M_{n+p}(c)$ be an $(n+p)$-dimensional complex space form with constant holomorphic sectional curvature $c$, complex structure $J$ and metric $g$. Let $M_n$ be an $n$-dimensional real submanifold immersed in $M_{n+p}(c)$ with the induced metric $g$. We denote by $T_x(M_n)$ and $\nu_x$ the tangent space and the normal space, respectively, of $M_n$ at $x$. $M_n$ is called the totally real submanifold of $M_{n+p}(c)$ if $\tilde{j}(T_x(M_n)) \subset \nu_x$.

Let $\sigma$ be the second fundamental form of the immersion, and $H$ the length of the mean curvature vector of $M_n$. For a normal vector $\xi$ on $M_n$, the tangential component $-A_\xi X$ of the covariant derivative $\nabla_x \xi$ satisfies $\tilde{g}(\sigma(X, Y), \xi) = g(A_\xi X, Y)$. We choose a local field of orthonormal frame $e_1, \ldots, e_n, e_{n+1}, \ldots, e_{n+p}$, $e_{n+1} = \tilde{e}_1, \ldots, e_{n+p} = \tilde{e}_n,$ $e_{n+1} = \tilde{e}_n, \ldots, e_{n+p} = \tilde{e}_n$ on $M_{n+p}(c)$ in such a way that, restricted to $M_n$, $e_1, \ldots, e_n$ are tangent to $M_n$. If we set $A_\alpha = A(e_\alpha) (\alpha = 0, 1, \ldots, n+p)$, then $\sigma(X, Y) = \sum g(A_\alpha X, Y) e_\alpha$.

Let $\tilde{R}$ and $R$ be the curvature tensor fields of $M_{n+p}(c)$ and $M_n$. Then

$$\tilde{R}(X, Y)Z = \frac{c}{4} \left( g(Y, Z)\tilde{Z} - g(X, Z)\tilde{Y} + g(Y, Z)\tilde{X} - g(X, Y)\tilde{Z} \right)$$

and the equation of Gauss is

$$\tilde{R}(X, Y; Z, W) = \tilde{R}(X, Y; Z, W) + \tilde{g}(\sigma(X, Z), \sigma(Y, W)) - \tilde{g}(\sigma(X, W), \sigma(Y, Z)),$$

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where \( \tilde{X}, \tilde{Y}, \tilde{Z} \) are vector fields on \( \tilde{M}_{n+p}(\tilde{c}) \), and \( X, Y, Z, W \) are vector fields on \( M_n \). Since \( M_n \) is totally real in \( M_{n+p}(\tilde{c}) \) we have

\[
R(X, Y; Z, W) = \frac{\tilde{c}}{4} \left\{ g(X, W)g(Y, Z) - g(X, Z)g(Y, W) \right\} + \sum \{ g(A_{\alpha}X, W)g(A_{\alpha}Y, Z) - g(A_{\alpha}X, Z)g(A_{\alpha}Y, W) \}.
\]

The sectional curvature \( K(X, Y) \) \( \{X, Y\} \) in \( K(X, Y) \) is supposed to be orthonormal and the Ricci tensor \( S(X, Y) \) of \( M_n \) are then given by

\[
K(X, Y) = \frac{\tilde{c}}{4} \sum \left\{ g(A_{\alpha}X, X)g(A_{\alpha}Y, Y) - g(A_{\alpha}X, Y)^2 \right\},
\]

\[
S(X, Y) = \frac{n-1}{2} \tilde{c} g(X, Y) + \sum (\text{tr } A_{\alpha})g(A_{\alpha}X, Y) - \sum g(A_{\alpha}X, A_{\alpha}Y).
\]

Let \( \rho \) be the scalar curvature of \( M_n \); then we have

\[
\rho = \sum_i S(e_i, e_i) = \frac{n(n-1)}{4} \tilde{c} + \sum (\text{tr } A_{\alpha})^2 - \|\sigma\|^2 = \frac{n(n-1)}{4} \tilde{c} + n^2 H^2 - \|\sigma\|^2;
\]

here \( \|\sigma\| \) is the length of the second fundamental form \( \sigma, \|\sigma\|^2 = \sum \text{tr } A_{\alpha}^2 \). If \( M \) is a minimal submanifold, then \( H = 0 \) and \( \rho = n(n-1)\tilde{c}/4 - \|\sigma\|^2 \).

We need the following algebraic lemma which is proved in [1].

**Lemma.** Let \( a_1, \ldots, a_n, b \) be \( n+1 \) (\( n \geq 2 \)) real numbers satisfying the following inequality:

\[
2 \left( \sum_{i=1}^{n} a_i \right)^2 \geq (n-1) \sum_{i=1}^{n} a_i^2 + b \quad (\text{resp. } > 1);
\]

then, for any distinct \( i, j; 1 \leq i < j \leq n \), we have \( 2a_ia_j \geq b/(n-1) \) (resp. >).

We now establish the

**Proposition.** Let \( M_n \) be a totally real submanifold of \( \tilde{M}_{n+p}(\tilde{c}) \). If the scalar curvature \( \rho \) of \( M_n \) satisfies

\[
\rho \geq n(n-1)/4 \cdot \tilde{c} + n^2(n-2)/(n-1) \cdot H^2 - a
\]

at a point \( p \), then every sectional curvature of \( M \) at \( p \) is \( \geq \tilde{c}/4 - a/2 \).

**Proof.** For the frame field, \( \{e_1, \ldots, e_{n+p}, e_{n+1}, \ldots, e_{(n+p)}\} \), chosen above let

\[
h_{ij}^\alpha = g(A_{\alpha}e_i, e_j);
\]
then \( A_\alpha = (h^\alpha_{ij}) \), \( A_\alpha \) is symmetric and
\[
\|\sigma\|^2 = \sum_{i,j} (h^\alpha_{ij})^2, \quad n^2 H^2 = \sum_\alpha \sum_i (h^\alpha_{ii})^2.
\]

Let \( \tau \) be any plane section of \( M_n \) spanned by two independent tangent vectors \( X, Y \) to \( M_n \). We choose the frame field suitably so that \( e_1, e_2 \) span \( \tau \), and that \( e_{n+1} \) is parallel to the mean curvature vector of \( M_n \). Then we have
\[
K(X, Y) = K(e_1, e_2) = \frac{\kappa}{4} + \sum_\alpha \{h^\alpha_{11}h^\alpha_{22} - (h^\alpha_{12})^2\},
\]
\[
n^2 H^2 = \left( \sum_i h^{n+1}_{ii} \right)^2.
\]

The assumption (*) is equivalent to \( \|\sigma\|^2 \leq n^2 H^2 / (n - 1) + a \). Hence we have
\[
\frac{1}{n-1} \left( \sum_i h^{n+1}_{ii} \right)^2 \geq \|\sigma\|^2 - a = \sum_i (h^{n+1}_{ii})^2 + \sum_{i \neq j} (h^{n+1}_{ij})^2 + \sum_{\alpha > n+1} (h^\alpha_{ij})^2 - a,
\]
\[
\left( \sum_i h^{n+1}_{ii} \right)^2 \geq (n - 1) \sum_i (h^{n+1}_{ii})^2 + \sum_{\alpha > n+1} (h^\alpha_{ii})^2 - a.
\]

By the Lemma we have
\[
2h^{n+1}_{11}h^{n+1}_{22} \geq \sum_{i \neq j} (h^{n+1}_{ij})^2 + \sum_{\alpha > n+1} (h^\alpha_{ij})^2 - a
\]
\[
\geq 2(h^{n+1}_{11})^2 + \sum_{\alpha > n+1} \{(h^\alpha_{11})^2 + (h^\alpha_{22})^2 + 2(h^\alpha_{12})^2\} - a
\]
\[
\geq 2(h^{n+1}_{12})^2 - 2 \sum_{\alpha > n+1} h^\alpha_{11}h^\alpha_{22} + 2 \sum_{\alpha > n+1} (h^\alpha_{12})^2 - a.
\]

Hence we obtain
\[
2 \sum_{\alpha > n+1} \{h^\alpha_{11}h^\alpha_{22} - (h^\alpha_{12})^2\} + a \geq 0.
\]

This yields
\[
K(X, Y) \geq \frac{\kappa}{4} - a/2.
\]

If \( M_n \) is minimal, then \( H = 0 \); the Proposition yields the

**Theorem.** Let \( M_n \) be a totally real minimal submanifold of \( \bar{M}_{n+p}(\kappa) \). If the scalar curvature \( \rho \) of \( M_n \) satisfies \( \rho \geq n(n - 1)\kappa/4 - a \) at a point \( p \), then every sectional curvature of \( M \) at \( p \) is \( \geq \kappa/4 - a/2 \).
REFERENCES


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