TRIPLES ON REFLECTIVE SUBCATEGORIES
OF FUNCTOR CATEGORIES

DAVID C. NEWELL

ABSTRACT. We show that if $S$ is a cocontinuous triple on a full reflective subcategory of a functor category then the category of $S$-algebras is again a full reflective subcategory of a functor category.

This note should be considered an addendum to [4], and definitions for all of the terminology and concepts we use can be found there.

We shall also fix $V$ to be a closed bicomplete category and, in this note, all of the category theory is done relative to $V$.

In [4], we have shown the following:

(I) If $C$ is a small category and $T$ is a cocontinuous triple on the functor category $V^C$, then there is a small category $C'$ and a functor $f: C \to C'$ so that

(a) $T$ is the triple induced by the adjoint pair $(f^*, f^!): V^{C'} \to V^C$, where $f^*: V^{C'} \to V^C$ is the functor induced by $f$ and $f^!$ is the left adjoint of $f^*$;

(b) the adjoint pair $(f^*, f^!)$ is tripleable, so that there is an equivalence of categories $V^{C'} \cong (V^C)^T$, where $(V^C)^T$ is the category of $T$-algebras.

(II) If $C$ is a small category and $T$ is any triple on $V^C$, there is a unique cocontinuous triple $\hat{T}$ on $V^C$ and a map of triples $\tau: \hat{T} \to T$ so that, if $R: C^\circ \to V^C$ denotes the right Yoneda embedding of $C^\circ$ into the representable functors of $V^C$, $\tau R$ is the identity (we shall refer to $\hat{T}$ as the cocontinuous approximation to $T$).

In this paper, we shall prove the following

1. Theorem. Suppose $C$ is a small category, $A$ is a full reflective subcategory of $V^C$ (i.e. the inclusion functor of $A$ to $V^C$ has a left adjoint) and $S$ is a cocontinuous triple on $A$. Then there is a small category...
ry $C'$ for which the category of $S$-algebras $A^S$ is a full reflective subcategory of $V^C'$.

The basic idea of the proof of this theorem is as follows: as we shall see, $S$ induces a triple $T$ on $V^C$ in an obvious way; we use (II) to construct the cocontinuous approximation $\hat{T}$ of $T$, and then we apply (I) to $\hat{T}$ to obtain the desired category $C'$.

The rest of this paper is devoted to showing that the above outline does indeed give a proof for the theorem.

Proof of the theorem. Let $A$ and $B$ be categories and suppose $(i, r): A \to B$ is an adjoint pair from $A$ to $B$ with unit $u: 1_A \to ir$ and counit $e: ri \to 1_A$. We shall let $R = (R, u, m)$ denote the triple on $B$ induced by $(i, r)$ (so that $m = ier$).

Suppose $S = (S, \eta', \mu')$ is a triple on $A$. Then $S$, together with $(i, r)$, induces a triple $T = (T, \eta, \mu)$ on $B$ by letting $T = isr$, $\eta = (inr') \cdot u$ and $\mu = (i\mu') \cdot (is er)$. Equivalently, $T$ is the triple induced by the adjoint pair obtained by composing the adjoint pair $(i, r): A \to B$ with the adjoint pair $(U^S, F^S): A^S \to A$, where $U^S: A^S \to A$ is the usual "underlying" functor from the category of $S$-algebras to $A$ and $F^S$ is the usual "free" functor.

One obtains easily the following facts:

1. the comparison functor $\sim: A^S \to B^T$;
2. there is a map of triples $\theta: R \to T$ given by $\theta = i\eta'r$.

2. Proposition. Suppose $A$ is a full reflective subcategory of $B$, i.e., the inclusion functor $i: A \to B$ has a left adjoint $r$, $S = (S, \eta', \mu')$ is a triple on $A$, and $T = (T, \eta, \mu)$ is the triple on $B$ induced by $S$ and $(i, r)$. Then

(a) the comparison functor $\sim: A^S \to B^T$ of (1) is an equivalence of categories, and

(b) if $R = (R, u, m)$ is the idempotent triple on $B$ induced by the adjoint pair $(i, r)$, then $T = TR = RT$ and the map of triples $\theta: R \to T = RT$ of (2) is given by $\theta = R\eta$. Furthermore, $\mu \cdot \theta T = \mu \cdot T\theta = 1_T$.

Proof. (a) Follows from Beck's tripleability theorem (see [7]). For (b), we have $RT = irisr = iS r = T$, as the counit $e: ri \to 1_A$ is the identity. Similarly $TR = T$. $R\eta = ir(i\eta'r \cdot u) = i\eta'r \cdot iru = i\eta'r$ (as $ri = 1_A$ and $ru = 1_r$) $\theta$. $\mu \cdot \theta T = i\mu'r \cdot is er \cdot i\eta'r isr = i\mu'r \cdot i\eta'Sr$ (as $e = 1$ and $ri = 1_A$) $i(\mu' \cdot \eta'Sr) = i1sr$ (as $S$ is a triple) $1_T$. Similarly $\mu \cdot T\theta = 1_T$. □

For the rest of this paper, let us make the following hypotheses.
(i) \( \mathcal{B} \) is cocomplete and there is a small category \( \mathcal{C} \) and a functor \( k: \mathcal{C} \to \mathcal{B} \) which is dense in \( \mathcal{B} \) (see [7]);

(ii) there is an adjoint pair \((i, r): \mathcal{A} \to \mathcal{B} \) whose counit is the identity (so that \( \mathcal{A} \) is equivalent to a full reflective subcategory of \( \mathcal{B} \)) and

\[
(*) \quad \mathcal{R} = (R, u, m) \text{ is the idempotent triple on } \mathcal{B} \text{ induced by } (i, r);
\]

(iii) \( \mathcal{S} = (S, \eta', \mu') \) is a cocontinuous triple on \( \mathcal{A} \) and, for \( \mathcal{T} = (T, \eta, \mu) \), the triple on \( \mathcal{B} \) induced by \( \mathcal{S} \) and \((i, r)\), there is a cocontinuous triple \( \hat{T} = (\hat{T}, \hat{\eta}, \hat{\mu}) \) on \( \mathcal{B} \) and a map of triples \( \tau: \hat{T} \to T \) with \( rk = 1 \).

We note that if \( \mathcal{C} \) is a small category, \( \mathcal{A} \) is a full reflective subcategory of \( \mathcal{V} \), and \( \mathcal{S} \) is a cocontinuous triple on \( \mathcal{A} \) (as in the hypotheses of 1), then \( \mathcal{B} = \mathcal{V} \), \( k \) the right Yoneda embedding \( \mathcal{R}: \mathcal{C} \to \mathcal{V} \), and \( \mathcal{T} \) the cocontinuous approximation of \( \mathcal{T} \) satisfy the above hypotheses (*)).

Recall that for \( X \) a category and for \( \mathcal{R} = (R, u, m) \) and \( \mathcal{T} = (T, \eta, \mu) \) two triples on \( X \), the composite triple of \( \mathcal{R} \) and \( \mathcal{T} \) is a triple \( \mathcal{R}T = (RT, u\eta, v) \) for which \( R\eta: R \to RT \) and \( uT: T \to RT \) are maps of triples and for which \( v \cdot (R\eta uT) = 1_{RT} \).

3. Proposition. Under the hypotheses (*)\( , \mathcal{T} \) is a composite triple of \( \mathcal{R} \) and \( \hat{T} \).

Proof. \( \tau \), being an adjoint, is cocontinuous. Since \( rT = riSr = Sr \) and since \( rr: rT \to rT = Sr \) is a natural transformation between cocontinuous functors for which \( rrk = 1 \), and since \( k \) is assumed dense, it follows that \( rr \) is an isomorphism of functors. Hence \( Rr: RT \to RT = T \) is an isomorphism of functors. Let \( \mathcal{RT} \) be \( R\hat{T} \) with the triple structure induced by that of \( T \) via the isomorphism \( R\tau \). Since \( \hat{\eta} : \tau = \eta \), one has \( (R\tau) \cdot (u\hat{\eta}) = u\eta = \eta \) so that \( u\hat{\eta} \) is the unit of \( \mathcal{RT} \).

\( R\hat{\eta}: R \to R\hat{T} \) is a map of triples, since \( R\tau \cdot R\hat{\eta} = R(\tau \cdot \hat{\eta}) = R\eta = \theta \) is a map of triples.

We now show that \( u\hat{T}: \hat{T} \to R\hat{T} \) is a map of triples. Now \( (R\tau \cdot \hat{T})k = Rrk \cdot u\hat{T}k = 1 \cdot uTk = uiSr = 1 \) (as \( ui = 1 \) and \( rk = 1 \)). Since \( \hat{T} \) is cocontinuous and \( k \) is dense, \( \hat{T} \) is the left Kan extension of \( \hat{T}k \) along \( k \) (see [7, p. 232]). The universal property of left Kan extensions gives us that \( R\tau \cdot n\hat{T} = \tau \), and since \( \tau \) is a map of triples, so is \( u\hat{T} \).

Finally, if \( v \) is the multiplication of \( R\hat{T} \) (induced by \( \mu \)), we have \( [v \cdot (R\hat{T}\mu)] = 1_{\hat{T}} \) since, by \( S2, \mu : \Omega T = 1 \), so that the diagram
4. Corollary. Under the hypotheses (*), $A^S$ is equivalent to a full reflective subcategory of $B^T$.

Proof. From [2, p. 122] and §3 we have a lifting of $R$ to a triple $\bar{R}$ on $B^T$ and an isomorphism of categories $\Phi: (B^T)\bar{R} \cong B^R$. But $R^T \cong T$ by §3 so that $B^R T \cong B T \cong A^S$ by §2. Since the underlying functor from $B^T$ to $B$ is faithful and $R$ is idempotent, the lifting $\bar{R}$ is idempotent. □

We note that Theorem 1 now follows from this corollary.

A problem arising from this theorem is the following: if $C$ is a small category, $A$ is a full reflective subcategory of $V^C$, and $S$ a cocontinuous triple on $A$, then is $A^S$ a full reflective subcategory of $V^C$ "of the same type"? For example, if $V = \text{Ab}$ (the category of abelian groups), $C$ a small abelian category, $L$ the category of left exact functors from $C$ to $\text{Ab}$, and $S$ a cocontinuous triple on $L$, then $L^S$ is a full reflective subcategory of $\text{Ab}^C$ for some preadditive category $C'$ by our theorem, but is $C'$ abelian and is $L^S$ the category of left exact functors from $C'$ to $\text{Ab}$?

The following is an example where the answer to this question is positive.

Let $C$ be a small category and let $J$ be a topology on $C$ making $(C, J)$ into a site (as in [5, Definition 1.2, pp. 256–303]). Let $A$ be the category of sheaves of sets on $C$, so that $A$ is a full reflective subcategory of the functor category $(i, r): A \to \text{Sets}^{C^\text{op}}$, where $i: A \to \text{Sets}^{C^\text{op}}$ is the inclusion functor, then $R$ is a left exact idempotent triple (where "left exact" means that the functor of $R$ preserves finite limits).

Now $\text{Sets}^{C^\text{op}}$ is an example of an elementary topos (as in [6, p. 5]) and one sees that the topologies $J$ on $C$ are in one-to-one correspondence with the topologies on the elementary topos $\text{Sets}^{C^\text{op}}$ (as defined in [6]). One can then show (using [6, Proposition 3.22, p. 70]) that the assignment
$J \mapsto \mathcal{R}$ as in the previous paragraph gives a one-to-one correspondence between topologies $J$ on $C$ and left exact idempotent triples $\mathcal{R}$ on $\text{Sets}^{\text{op}}$.

5. **Theorem.** Let $C$ be a small category, $J$ a topology on $C$, $A$ the category of sheaves of sets on $C$ with respect to $J$, and $S$ a cocontinuous triple on $A$. Then there is a small category $C'$ and a topology $J'$ on $C'$ so that $A^S$ is a category of sheaves of sets on $C'$ with respect to $J'$.

**Proof.** Let $B = \text{Sets}^{\text{op}}$, $R$ the left exact idempotent triple corresponding to $J$, $T$ the triple on $B$ induced by $S$, and $\hat{T}$ the cocontinuous approximation to $T$. Let $C'$ be a category for which $B^{\hat{T}} \cong \text{Sets}^{\text{op}}$ (as in I). We have that $A^S \cong (B^{\hat{T}})^{\mathcal{R}}$, where $\mathcal{R}$ is a lifting of $R$. Now the underlying functor from $B^{\hat{T}}$ to $B$ is not only faithful but preserves and creates limits. Therefore, since $R$ is a left exact idempotent triple, $\mathcal{R}$ must also be. We now let $J'$ be the topology on $C'$ corresponding to $\mathcal{R}$, and we are done. $\square$

The referees of this paper have pointed out that Theorem 5 follows from Giraud's theorem (see [3, pp. 108–109]) in the following way. Since $S$ is cocontinuous, the underlying functor $U: A^S \rightarrow A$ creates both limits and colimits. From this one sees that $A^S$ is an exact category with limits, colimits, and disjoint universal sums. The free algebras in $A^S$ on the set of generators in $A$ are easily seen to form a set of generators for $A^S$. Thus, by Giraud's theorem, $A^S$ is a topos, from which our Theorem 5 follows.

**REFERENCES**


DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CALIFORNIA, IRVINE, CALIFORNIA 92664