A PROOF OF BERNSTEIN'S THEOREM
ON REGULARLY MONOTONIC FUNCTIONS

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ABSTRACT. A function is called "regularly monotone" if it is of class $C^\infty$ and each derivative is of a fixed sign (which may depend on the order of the derivative). We present a short proof of Bernstein's theorem on the analyticity of such functions.

This paper presents a short proof of Bernstein's theorem [1] on "regularly monotone" functions. For background on this subject (and a presentation of Bernstein's original proof) we refer the reader to the brief survey paper by Boas [2], and the references cited there. The book [3] contains a proof of a special case of Bernstein's theorem. That proof for that special case partially motivated the proof we present here. Before giving our proof of Bernstein's theorem, we recall that a regularly monotone function is a function of class $C^\infty$ on a real interval $(a, b)$ for which each derivative is of fixed sign on $(a, b)$.

Theorem. If $F(x)$ is regularly monotone on $(a, b)$, then $F(x)$ is analytic on $(a, b)$.

Proof. We assume for convenience that $a = -b < 0$. We will prove that the even part of $F(x)$, $f(x)$, is analytic at $x = 0$. An analogous proof holds for the odd part of $F(x)$, and the analyticity of $F(x)$ thereby follows. We begin by noting that $f(x)$ is itself regularly monotone on $0 < x < b$. The following lemma is strategic for our proof.

Lemma. If $f^n(x) \leq 0$, $f^{n+1}(x) \geq 0$, and $f^{n+2}(x) \geq 0$, then for $x \in [0, b)$, $|R_n(x)| \geq |R_{n+1}(x)|$, where $R_m(x)$ is the $m$th remainder in the Taylor expansion of $f$ about $x = 0$.

Proof of Lemma. Write the remainder as

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(1) \[ R_n(x) = \frac{1}{(n-1)!} \int_0^x f^{(n)}(t)(x-t)^{n-1} dt. \]

A sufficient condition for the Lemma to be true is that

(2) \[ -f^{(n)}(t) - \frac{(x-t)}{n} f^{(n+1)}(t) \geq 0 \]

for \( t \) on \((0, x)\). Since \( f^{(n+1)}(t) \geq 0 \), replacing \( n \) by 1 in (2) yields a sufficient condition for (2) to be true; namely

\[ -f^{(n)}(t) - (x-t)f^{(n+1)}(t) \geq 0, \]

or, if we write \( g \) for \( f^{(n)} \),

(3) \[ -g(t) - (x-t)g'(t) \geq 0. \]

By Rolle's theorem there exists a \( \xi \) on \((t, x)\) such that

(4) \[ -g(\xi) - (x-\xi)g'(\xi) = -g(x) \geq 0. \]

Since \( g'' = f^{(n+2)} \geq 0 \), (3) follows from (4) (Q.E.D. Lemma).

If instead of the sign sequence of the Lemma one has either of the sign triples +, +, + or +, +, −, the Lemma's conclusion is immediate. Only for polynomials (analytic functions) can the triple +, −, + occur when \( f \) is even. For, in this case, \( f^{(n)} \) is even or odd. If \( f^{(n)} \) is even, say, then \( f^{(n+1)}(0) = 0 \). By supposition, \( f^{(n+2)}(x) \geq 0 \) for \( x \geq 0 \). Thus \( f^{(n+1)}(x) \geq 0 \) for \( x \geq 0 \). But also by supposition, \( f^{(n+1)}(x) \leq 0 \) for \( x \geq 0 \). Thus \( f^{(n+1)}(x) \equiv 0 \Rightarrow f \) is a polynomial. An analogous proof holds for \( f^{(n)} \) odd. Since one of these sign triples is always attainable (perhaps after \( f \to -f \)), it follows that for \( f \) even

(5) \[ |R_n(x)| \geq |R_{n+1}(x)| \quad \text{for } n = 0, 1, 2, \ldots, \]

for \( x \) on \([0, b)\).

Frequently \( f^n \) and \( f^{n+1} \) are both \( \geq 0 \) (or both \( \leq 0 \)) (else eventually sign \( (f^{(n)}) = (-1)^{n+K} \Rightarrow f \) is polynomial). Rewriting (1) as

(6) \[ R_n(x) = \frac{x^n}{(n-1)!} \int_0^1 f^{(n)}(xt)(1-t)^{n-1} dt, \]

then for, say, \( f^{(n)} \) and \( f^{(n+1)} \geq 0 \) and \( x \geq 0 \), one has

(7) \[ 0 \leq R_n(x) \leq \frac{x^n}{(n-1)!} \int_0^1 f^{(n)}(b't)(1-t)^{n-1} dt, \]

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where \( x < b' < b \), since \( f^{(n_i)} \) is nondecreasing. Thus

\[
0 \leq R_{n_i}(x) \leq (x/b')^n R_{n_i}(b').
\]

But \( R_n(b') \) is bounded, whence \( R_n(x) \to 0 \) as \( n \to \infty \). Since for \( f(x) \) even, \( R_n(x) = R_n(-x) \), this proves the analyticity of \( f(x) \), the even part of \( F(x) \). The analyticity of \( F(x) \) itself then follows as initially indicated.

**REFERENCES**


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