CONVERGENCE SETS IN REFLEXIVE BANACH SPACES

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ABSTRACT. A closed linear subspace $M$ of a reflexive Banach space $X$ with $X$ and $X^*$ strictly convex is the range of a linear contractive projection iff $J(M)$ is a linear subspace of $X^*$. Hence the convergence set of a net of linear contractions is the range of a contractive projection if $X$ and $X^*$ are locally uniformly convex.

Let $X$ be a Banach space over $\mathbb{C}$ or $\mathbb{R}$, and let $(T_n)$ be a net of linear contractions on $X$. The convergence set for $(T_n)$ is $\{x \in X: T_n x \to x\}$. Bernau [1] showed that if $X$ is an $L_p$ space, $p \in (1, \infty)$, then a convergence set is the range of a linear contractive projection. A simplification and generalization of this result follows from the characterisation of ranges of contractive linear projections of Theorem 1, and is given as Theorem 2 below.

Let $S$ be a subset of $X$. Then the shadow of $S$ is the set of $x$ in $X$ such that $T_n x \to x$ for every net of linear contractions on $X$ such that $T_n y \to y$ for all $y$ in $S$. Assuming $X$ to be an $L_p$ space, Bernau [1] showed that the shadow of $S$ is the range of a contractive projection, and that if $E$ is the range of a contractive projection, and $E$ contains $S$, then $E$ contains the shadow of $S$. This result holds generally, and is given as Corollary 2.

Theorem 3 considers finding the projection in terms of the net $(T_n)$.

By a nearest point projection (on a subset $K$ of a Banach space $X$) we mean a function $P$ taking $x$ in $X$ to a nearest point in $K$.

Lemma 1. A set is the range of a linear contractive projection iff it is the nullspace of a linear nearest point projection.

Proof. $Q$ is a linear nearest point projection iff $1 - Q$ is a linear contractive projection.

Theorem 1. Let $X$ be a strictly convex reflexive Banach space with strictly convex dual $X^*$. Let $J: X \to X^*$ be the duality map; $\|Jx\| = \|x\|$, $(Jx, x) = \|x\|^2$. Then a closed linear subspace $M$ of $X$ is the range of a
linear contractive projection iff $J(M)$ is a linear subspace of $X^*$.

Proof. Suppose $J(M)$ is linear. Let $Q$ be the nearest point projection on $J(M)$. (There exist nearest points since $X$ is reflexive, and only one since $X$ is strictly convex.) For $x \in X$, $Qx \in J(M)$ is defined by the property that for $y \in J(M)$, $(J(x - Qx), y) = 0$. (The real part of $J$ is the Gâteaux derivative of the function taking $x$ to $\|x\|^2/2$, since $X^*$ is strictly convex.) $J(M)$ is closed since $J^{-1}$ is continuous from the strong to the weak topology (since $X$ is reflexive and strictly convex). Hence, $Qx$ is defined by $Qx \in (J(M))^\perp$ and $J(x - Qx) \in J(M)$, or $x - Qx \in M$. This shows $Q$ is linear, for if $y \in X$, $y - Qy \in M$, $Qy \in (J(M))^\perp$, then $(x + y) - (Qx + Qy) \in M$, and $Qx + Qy \in (J(M))^\perp$, giving $Q(x + y) = Qx + Qy$, and similarly $Q(\alpha x) = \alpha Q(x)$.

Since $Qx = 0$ iff $x \in M$ the result follows from Lemma 1.

Conversely, suppose $M = R(P)$, the range of a contractive linear projection. If $m \in M$,

$$\|P^*jm\| \leq \|jm\| \leq \|m\|, \quad \text{and} \quad (P^*jm, m) = (Jm, Pm) = \|m\|^2,$$

giving $P^*jm = Jm$, since $X^*$ is strictly convex. Hence, $J(M) \subset R(P^*)$. Replacing $P$ by $P^*$ and $J$ by $J^{-1}$ (since $X$ is strictly convex), $J^{-1}R(P^*) \subset M$, giving $J(M) = R(P^*)$, completing the proof.

Corollary 0. Let $X$ be a reflexive Banach lattice with $X$ and $X^*$ strictly convex. Then a closed subspace $M$ is the range of a positive linear contractive projection iff $JM$ is a linear subspace and sublattice of $X^*$ iff $JM$ is a linear subspace and $M$ is a sublattice.

Proof. Let $P$ be a positive linear contractive projection. Since $P$ is positive, for $x$ in $X$, $P(x^+) \geq (Px)^+$. Replacing $x$ by $Px$ gives $P((Px)^+) \geq (Px)^+$. Let $y$ be a convex combination of $P((Px)^+)$ and $(Px)^+$. Since $X$ is a Banach lattice, $\|y\| \geq \|(Px)^+\|$. Since $\|P\| = 1$, the opposite inequality holds. By strict convexity, $P((Px)^+) = (Px)^+$, which implies $M$ is a sublattice. The argument applied to $P^*$ gives $J(M)$ a sublattice.

Suppose $JM$ is a linear subspace and $M$ is a sublattice. For $x$ in $M$ and $y$ in $J(M)^\perp$, $x^+ \in M$, giving

$$\|x^+\|^2 = (J(x^+), x) = (J(x^+), x + y) \leq (J(x^+), (x + y)^+) \leq \|x^+\| \|(x + y)^+\|.$$

If $x + y \leq 0$, then $x \leq 0$. Since $X = M + J(M)^\perp$, the linear contractive projection on $M$ is positive.

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Corollary 1. Let $X$ be as in Theorem 1. Let $(M_i)_{i \in I}$ be a net of ranges of linear contractive projections. Then $M = \bigcup_{i \in I} M_i$ is the range of a linear contractive projection.

Proof. $J(\bigcup_{i \in I} M_i) = \bigcup_{i \in I} J(M_i)$ is a linear subspace since each $J(M_i)$ is.

Theorem 2. Let $X$ be a reflexive Banach space, with $X$ and $X^*$ locally uniformly convex. Then convergence sets are ranges of linear contractive projections.

Proof. Let $(T_n)$ be a net of contractions with convergence set $M$. If $m \in M$, $(T_n^* T_m, m) \to \|m\|^2$, and $\|T_n^* T_m\| \leq \|m\|$, giving $T_n^* T_m \to T_m$ since $X$ is locally uniformly convex. (If $X$ is reflexive and locally uniformly convex, then given $y \neq 0$, and $\epsilon > 0$, there exists $\delta > 0$ such that if $\|x\| \leq \|y\|$, and $\|x - y\| \geq \epsilon$, then $\|(x + y)/2\| \leq \|y\|(1 - \delta)$. Hence if a subsequence $T_n^* T_m \to y$ weakly, then $\|y\| = \|m\|$. Since $\|T_n^* T_m\| \to \|y\|$, $\|(T_n^* T_m + y)/2\| \to \|y\|$, which implies $T_n^* T_m \to y$. Since $y$ satisfies the inequalities defining $T_m$, $y = T_m$. Hence, $M^* \supset J(M)$ where $M^* = \{f \in X^*: T_n^* f \to f\}$. Similarly, $M \supset J^{-1}(M^*)$, giving equality. The result follows by Theorem 1.

Corollary 2. Let $X$ be as in Theorem 2. Let $S \subset X$. Then the shadow of $S$ is the smallest convergence set containing $S$.

Proof. By definition, the shadow of $S$ is the intersection of all convergence sets containing $S$, which is a convergence set by Theorem 2 and Corollary 1.

Corollary 3. Let $X$ be as in Theorem 1 and $(T_n)$ a net of linear contractions. Then $\{x: T_n^* x \to x\}$, the weak convergence set, is the range of a linear contractive projection.

Proof. By the proof of Theorem 2.

Lemma 2. Let $X$ be a reflexive Banach space, with $X$ and $X^*$ strictly convex. If $M$ is the range of a linear contractive projection $P$, it is the range of only one.

Proof. By Theorem 1, $I - P$ is the nearest point projection on $N(P) = R(P^*)^\perp = (JM)^\perp$.

Lemma 3. Let $X$ be a reflexive Banach space with $X$ and $X^*$ strictly convex. Given a finite set $(T_n)_{n \in F}$ of linear contractions, then the linear contractive projection on $\{x: T_n^* x = x$ for $n \in F\}$ is $\lim_{p \to \infty} \lim_{k \to \infty} A^F_p k$.
in the strong operator topology), where
\[
A^F_{p,k} = \frac{1}{p} \sum_{j=0}^{p-1} \left( \prod_{n \in F} \frac{1}{k} \sum_{i=0}^{k-1} T_i^n \right)^j,
\]
and the product can be taken in any order.

**Proof.** For \( n \) in \( F \), let \( P_n \) be the linear contractive projection on \( N(I - T_n) \), unique by Lemma 2. By the mean ergodic theorem, \( T_n(k) = (1/k) \sum_{i=0}^{k-1} T_i^n \) converges to \( P_n \).

By induction we show \( \prod_{n \in F} P_n x = x \), where \( I \subseteq F \), implies \( P_n x = x \) for \( n \) in \( I \). Suppose it is true for \( m \) elements in \( I \); then for \( n \) in \( F \), suppose \( P_n \prod_{i \in I} P_i x = x \). If \( y \) is a convex linear combination of \( x \) and \( \prod_{i \in I} P_i x \), then \( P_n y = x \), giving \( \|y\| = \|x\| \). By strict convexity, \( x = \prod_{i \in I} P_i x \), giving \( P_n x = x \) for \( x \) in \( I \) by the inductive hypothesis, and hence \( P_n x = x \). Hence,
\[
\bigcap_{n \in F} N(I - T_n) = \bigcap_n N(I - P_n) = N\left( I - \prod_{n \in F} P_n \right).
\]

By the mean ergodic theorem, the nonexpansive projection on this set is \( \lim_{p \to +\infty} (1/p) \sum_{j=0}^{p-1} (I P_n)^j \). By continuity of multiplication of operators in the strong topology, we can take the limits outside, giving the formula.

**Lemma 4.** Let \( X \) be a reflexive Banach space, and \( A = (A_n)_{n \in S} \) a net of bounded linear operators on \( X \), \( \|A_n\| \leq M \) for all \( n \). Define \( N(A) = \{x: A_n x \to 0\} \) and \( R(A) = \{y: \text{there exists a subnet } (A_{m(m)})_{m \in T} \text{ of } A, \) a bounded set of \( X \), \( \|y_{m(m)}\|: m \in T \}, \) and for \( N \) in \( T \) there is a set of positive numbers \( \alpha_m^N \) for finitely many \( m \geq N \) in \( T \), \( \sum_m \alpha_m^N = 1 \), and \( y = \lim_{N} \sum_m \alpha_m^N A_{m(m)} y_{m(m)} \).

Then by defining \( A^* = (A^*_n) \), we have \( R(A)^\perp = N(A^*) \).

**Proof.** Take \( f \in N(A^*) \), \( y \in R(A) \), \( y = \lim_N \sum_m \alpha_m^N A_{m(m)} y_{m(m)} \), where \( \|y_{m(m)}\| \leq K \) for all \( m \). Then
\[
(f, y) = \lim_N \langle f, \sum_m \alpha_m^N A_{m(m)} y_{m(m)} \rangle = \lim_N \langle A^*_n(m) f, \sum_m \alpha_m^N y_{m(m)} \rangle 
\]
\[
\leq \lim \|A^*_n(m) f\|K = 0.
\]
Suppose instead that \( f \in R(A)^\perp \). We wish to show that if \( T \) is a cofinal subset of \( S \), then \( (A^*_n f)_{n \in T} \) has a subnet converging to zero. Take \( y_n =
$J^{-1}A^*_n$ for $n$ in $T$. By weak compactness, there is a weak cluster point $y$ for $(A_n y_n)_{n \in T}$. For $N$ in $T$, $y$ is in the weak closure of $\{A_p y_p : p \in T, p \geq N\}$, and hence the strong closure of its convex hull. Thus for $U$ a neighborhood of $y$, we can take $\alpha^N, U \geq 0$, for $p \geq N$, $p$ in $T$, nonzero for only finitely many $p$, $\sum_p \alpha^N_p A_p y_p \in U$. Let $Q$ be the directed set of neighborhoods of $y$; then for $(p, U)$ in $T \times Q$, putting $A_{(p, U)} = A_p$ gives $(A_{(p, U)})_{(p, U) \in T \times Q}$ a subnet of $(A_p)_{p \in S}$ and

$$y = \lim_{(N, U) \in T \times Q} \sum_p \alpha^N_p A_p y_p,$$

giving $y \in R(A)$. But

$$0 = \lim_{(N, U)} \left( f, \sum_p \alpha^N_p A_p y_p \right) = \lim_{(N, U)} \sum_p \alpha^N_p \|A^{*}f\|^2 \geq \lim_{p \in T} \|A^*_p f\|^2,$$

completing the proof.

**Theorem 3.** Let $X$ be a reflexive Banach space with $X$ and $X^*$ strictly convex. Let $(T_n)_{n \in S}$ be a net of contractions such that $x = \lim T_n x$ implies $x = T_n x$ eventually. Then the convergence set $M$ is the range of the linear contractive projection

$$A = \lim_{N \in Q} \lim_{p \rightarrow \infty} \lim_{k \rightarrow \infty} A_{p, k},$$

where $Q_N$ is the set of finite subsets of the set of elements $p$ of $S$, $p \geq N$, directed under inclusion. $A_{p, k}$ is defined in Lemma 3.

**Proof.** $M = \bigcup_{N \in S} \bigcap_{n \geq N} R(I - T_n)$. By Corollary 1, $M$ is the range of a linear contractive projection. Set $I - T = (I - T_n)_{n \in S}$. By Lemma 4, $X = M + \text{cl} R(I - T)$. Given $\epsilon > 0$, for $x$ in $M$ and $z$ in $\text{cl} R(I - T)$, take $y$ in $R(I - T)$, $\|y - z\| < \epsilon/3$, let $y = \lim_{N} \sum_m \alpha^N_m (I - T_n) y_n(m)$, where $\|y_n(m)\| \leq K$, take $N$, $\alpha^N_m$, such that $x \in N(I - T_n)$ for $n \geq N$ and $\|y - \Sigma \alpha^N_m (I - T_n) y_n(m)\| < \epsilon/3$. Take $F_n$ the support of $\alpha^N_m$, $F = n(F_N)$, and set

$$A_{p, k}^{F} = \frac{1}{p} \sum_{j=0}^{p-1} \left( \prod_{n \in F} T_n(k) \right)^j,$$

where $T_n(k) = (1/k) \sum_{i=0}^{k-1} T_n^i$, and some order is chosen in the product. Choose $p$ and $q$ by Lemma 3 so that $\|A_{p, k}^{F} \Sigma \alpha^N_m (I - T_n) y_n(m)\| < \epsilon/3$.

Then
\[ \|A_{p,k}^F(x + z) - x\| \leq \|A_{p,k}^F(z - y)\| + \|A_{p,k}^F(y - \sum \alpha_m^N(I - T_{n(m)})y_{n(m)})\| \\
+ \|A_{p,k}^F \sum \alpha_m^N(I - T_{n(m)})y_{n(m)}\| \\
\leq \epsilon/3 + \epsilon/3 + \epsilon/3, \]

proving the claim.

REFERENCE


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