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SHORTER NOTES

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ON SEQUENCES SPANNING A COMPLEX l_1 SPACE

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ABSTRACT. If (f_n) is a bounded sequence in a complex Banach space B, and no subsequence of (f_n) is weakly Cauchy, then a subsequence of (f_n) is equivalent to the unit vector basis of the complex l_1 space.

In a recent paper [1], H. P. Rosenthal proves that if a bounded sequence (f_n) of elements of a real Banach space has no weak Cauchy subsequence, then a subsequence of (f_n) is equivalent to the unit vector basis of real l_1. The purpose of the present note is to prove the analogous result for complex Banach spaces. The proof follows the lines of [1]. *Any* $M \subseteq \mathbf{N}$ *is assumed to be infinite*, and $(f_n)_{n \in M}$ has the natural meaning (as a sequence).

Let S be the unit ball of B^*, (f_n) as in the abstract, i.e. a sequence of uniformly bounded affine complex functions on S with no pointwise convergent subsequence. Let $\mathfrak{D}$ be the set of all pairs (D_1, D_2) of open "rational" discs in the complex plane $\mathbf{C}$ such that $\operatorname{diam}(D_1) = \operatorname{diam}(D_2) < \frac{1}{2} \operatorname{dist}(D_1, D_2)$, and let $\{(D_1^k, D_2^k); k \in \mathbf{N}\}$ be an enumeration of $\mathfrak{D}$.

Claim. There are $k_0 \in \mathbf{N}$ *and a subset* $M \subseteq \mathbf{N}$ *so that for every* $L \subseteq M$ *there is* $s \in S$ *s.t. the sequence* $(f_n(s))_{n \in L}$ *has accumulation points both in* $D_1^{k_0}$ *and in* $D_2^{k_0}$.

Otherwise, we can construct $\mathbf{N} \supseteq M_1 \supseteq M_2 \supseteq \cdots$ so that for any $k \in \mathbf{N}$ and any $s \in S$ the sequence $(f_n(s))_{n \in M_k}$ has all its accumulation points outside D_1^k or all of them outside D_2^k. Choose $m_1 < m_2 < \cdots$ s.t. $m_k \in M_k$

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$(k \in \mathbf{N})$, and let $L = \{m_k;\ k \in \mathbf{N}\}$. $(f_n)_{n \in L}$ does not converge pointwise on S so there are $s \in S$ and $p \neq q$ in $\mathbf{C}$ so that p and q are accumulation points of $(f_n(s))_{n \in L}$. Clearly, there is $k \in \mathbf{N}$ s.t. $p \in D_1^k$, $q \in D_2^k$, but then $(f_n(s))_{n \in M_k}$ has accumulation points in both D_1^k and D_2^k.

Now let k_0 and M be as above, let α be the center of $D_1^{k_0}$, β that of $D_2^{k_0}$, and let $2\delta = \operatorname{dist}(D_1^{k_0}, D_2^{k_0})$. Multiplying all the functions f_n by $|\beta - \alpha|/(\beta - \alpha)$, we may assume that $\beta - \alpha > 0$. For $n \in M$, let $A_n = \{s \in S;\ f_n(s) \in D_1^{k_0}\}$, and $B_n = \{s \in S;\ f_n(s) \in D_2^{k_0}\}$. Using Theorem 2 of [1], we get $L \subseteq M$ so that for any pair of disjoint finite sets $E, F \subseteq L$, $\bigcap_{n \in E} A_n \cap \bigcap_{n \in F} B_n \neq \emptyset$.

The sequence $(f_n)_{n \in L}$ *is equivalent over the complex scalars to the unit vector basis of* l_1: It is enough to show that $\|\sum_{k \in E} c_k f_k\| \geq (\delta/4) \sum_{k \in E} |c_k|$, for any finite $E \subseteq L$ and any choice of $c_k = a_k + i b_k (k \in E)$. Without loss of generality, $\sum_{k \in E} |a_k| \geq \sum_{k \in E} |b_k|$. Choose $s \in \bigcap_{a_k \geq 0} B_k \cap \bigcap_{a_k < 0} A_k$, and $t \in \bigcap_{a_k \geq 0} A_k \cap \bigcap_{a_k < 0} B_k$. Thus,

$$\left\| \sum_{k \in E} c_k f_k \right\| \geq \operatorname{Re} \sum_{k \in E} c_k f_k \left(\frac{s - t}{2} \right)$$

$$\geq \sum_{k \in E} a_k \operatorname{Re} f_k \left(\frac{s - t}{2} \right) - \sum_{k \in E} \left| b_k \operatorname{Im} f_k \left(\frac{s - t}{2} \right) \right|$$

$$\geq \delta \sum_{k \in E} |a_k| - \frac{\delta}{2} \sum_{k \in E} |b_k| \geq \frac{\delta}{2} \sum_{k \in E} |a_k| \geq \frac{\delta}{4} \sum_{k \in E} |c_k|.$$

The third inequality follows from the geometrically clear fact that $u \in D_1^{k_0}$, $v \in D_2^{k_0}$ imply:

$$\operatorname{Re}(v - u) \geq 2\delta \quad \text{and} \quad \operatorname{Im}(v - u) \leq \operatorname{diam}(D_1^{k_0}) < \delta.$$

REFERENCE

1. H. P. Rosenthal, *A characterization of Banach spaces containing* l_1, Proc. Nat. Acad. Sci. 71 (1974), 2411–2413.

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