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HELSON SETS WHICH DISOBEY SPECTRAL SYNTHESIS

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ABSTRACT. In this paper it is shown that every nondiscrete LCA group contains a compact independent Helson set which disobeys spectral synthesis.

In [3] T. W. Körner has constructed an independent compact H_1-set of type M. This result solves negatively the long-standing problem whether or not every Helson set obeys spectral synthesis. Körner's construction of such a set is, however, very complicated, and R. Kaufman [2] has simplified it (see also [6]. In this paper, we modify Kaufman's method to prove that every nondiscrete metrizable LCA group contains an independent compact Helson set of type M. Consequently it is shown that every nondiscrete LCA group contains a Helson set which disobeys spectral synthesis.

Let G be a LCA group with dual $\hat{G}$. We denote by $A(G)$ and $PM(G)$ the Fourier algebra on G and the conjugate space of $A(G)$, respectively. Each element of $PM(G)$ is called a pseudo-measure on G. For $f \in A(G)$ and $P \in PM(G)$, we define

$$\langle f, P\rangle = (\hat{f} * \hat{P})(1) = \int_{\hat{G}} \hat{f}(\chi)\hat{P}(\chi^{-1})\,d\chi,$$

where $\hat{f}$ and $\hat{P}$ denote the functions in $L^1(\hat{G})$ and in $L^\infty(\hat{G})$ whose (inverse) Fourier transforms are f and P, respectively. If $\mu \in M(\hat{G})$ and $\check{\mu}(x) = \int_{\hat{G}} \chi(x)\,d\mu(\chi)$ $(x \in G)$, then we denote by $\check{\mu}P$ the pseudo-measure on G defined by the requirement $(\check{\mu}P)\hat{} = \mu * \hat{P}$. It is well known that if $P \in PM(G)$ has compact support, then $\hat{P}$ can be chosen from the space $C(G)$; we will always do this. For such a P, $\langle \check{\mu}, P\rangle = (\mu * \hat{P})(1)$ $(\mu \in M(\hat{G}))$ is well defined, and we have $\langle \chi, \gamma P\rangle = \hat{P}(\chi^{-1}\gamma^{-1})$ $(\chi, \gamma \in \hat{G})$. A pseudo-function on G is any pseudo-measure whose Fourier transform is a continuous function on $\hat{G}$ which vanishes at infinity. The space of all pseudo-functions is denoted by $PF(G)$.

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Let now E be a closed set in G, and $q = q(G)$ the supremum of the natural numbers n such that every neighborhood of 0_G contains an element with order $\geq n$. E is called an M-set (or a set of type M) if $PF(E) \neq 0$ (that is, if E carries a nonzero pseudo-function); strongly independent if E is independent in the usual sense and $\mathrm{ord}(e) = q$ for all $e \in E$; and an H_1-set if $\|\mu\|_M = \|\mu\|_{PM}$ for all $\mu \in M(E)$. In the case $q < \infty$, E is called a K_q-set if every function $f \in C(E)$ with $f^q = 1$ is the restriction of a character in $\hat{G}$; and a weak K_q-set if to each $\mu \in M^+(E)$ and $\epsilon > 0$ there corresponds a K_q-set $K \subset E$ such that $\mu(E \setminus K) < \epsilon$.

Our main result now follows (cf. [3, p. 105]).

Theorem 1. *Every nondiscrete LCA group contains a strongly independent compact set which is either an H_1-set or a weak K_q-set but disobeys spectral synthesis. If, in addition, the group under consideration is metrizable, such a set can be chosen as a set of type M.*

The proof becomes clear after some lemmas and theorems are established.

Lemma 1. *To each $0 < \epsilon < 1$ there corresponds a natural number N_ϵ with the following property: for any neighborhood V of 0 of a compact abelian group H and any natural number $N \geq N_\epsilon$, there exists an $F \in A(H^N)$ such that*

(i) $\mathrm{supp}\, F \subset V_\epsilon^N$, *and*

(ii) $|\hat{F}(k)| < \epsilon |\hat{F}(0)|$ $\forall k \in \hat{H}^N \setminus \{0\}$.

Here V_ϵ^N is the set of all $(x_j)_1^N \in H^N$ such that $x_j \in V$ for at least $(1-\epsilon)N$ indices $j = 1, 2, \cdots, N$, and $\hat{H}^N$ denotes the additively written dual of H^N.

The proof is essentially identical with that of [2, Lemma 2], so we omit it.

Lemma 2. *Let G be a LCA I-group, $\hat{K}$ a compact subset of $\hat{G}$, and M, $N \in \mathbf{N}$ (the natural numbers). Then there exist N characters $\chi_1, \cdots, \chi_N \in \hat{G}$ such that the sets $\chi_1^{k_1} \cdots \chi_N^{k_N} \hat{K}$ $(k_j \in \{0, \pm 1, \cdots, \pm M\}, 1 \leq j \leq N)$ are pairwise disjoint.*

Proof. By the structure theorem [1, (9.8)], G contains an open subgroup G_0 which is topologically isomorphic to $R^a \times H$ for some $a \in \{0, 1, 2, \cdots\}$ and some compact abelian group H. Since there exists a continuous homomorphism of $\hat{G}$ onto $\hat{G}_0$, we may assume that $G = G_0 = R^a \times H$.

Suppose that there exists a $\gamma \in \hat{G}$ such that

(1) $$\lim_n \gamma^n = \infty.$$

Take $p \in N$ so large that $n > p \quad \gamma^n \notin \hat{K}^{-1}\hat{K}$, and choose N natural numbers $q_1, \cdots, q_N$ so that $|k_1 q_1 + \cdots + k_N q_N| > p$ for all choices of $k_j \in \{0, \pm 1, \pm 2, \cdots, \pm 2M\}$, $1 \leq j \leq N$, such that $(k_1, \cdots, k_N) \neq (0, \cdots, 0)$. Then the elements $\chi_j = \gamma^{q_j}$, $1 \leq j \leq N$, have the required property.

Assume now that no $\gamma \in \hat{G}$ satisfies (1). Then $a = 0$, $G = H$ is compact and $\hat{G}$ is a torsion group. Therefore every finitely-generated subgroup of $\hat{G}$ is finite. But $\hat{G}$ is not of bounded order because G is an I-group (see [4, 2.5.4]). Thus, setting $\chi_0 = 1$, we can find $\chi_1, \cdots, \chi_N \in \hat{G}$ so that

$$\operatorname{ord}(\chi_j) > 2M \cdot \operatorname{Card}[G_p(\hat{K} \cup \{\chi_0, \cdots, \chi_{j-1}\})]$$

for all $j = 1, 2, \cdots, N$. As is easily seen, the χ_j, $1 \leq j \leq N$, have the required property.

Lemma 3. *Let G be a LCA I-group, and τ a pseudo-function on G whose support E is compact. Let also $0 < \epsilon < 1$, and g any function in $C(E)$ whose range is a finite subset of $T = \{z: |z| = 1\}$. Then there exist $\tau' \in PF(G)$ and $\chi_j \in \hat{G}$, $1 \leq j \leq N$, such that*

(a) $\|\tau' - \tau\|_{PM} < \epsilon$,

(b) $\operatorname{supp} \tau' \subset \operatorname{supp} \tau$,

(c) $\|g - (1/N) \sum_{j=1}^N \chi_j\|_{C(\operatorname{supp} \tau')} < \epsilon$.

Proof. We may assume that g is defined and continuous on some compact neighborhood U of E, and that $g(U)$ is a finite subset of T. Thus

(1) $$C = \sup\{\|g^k\|_{A(U)}: k \in \mathbf{Z}\} < \infty,$$

where

$$\|g^k\|_{A(U)} = \inf\{\|f\|_{A(G)}: f \in A(G), f = g^k \text{ on } U\}.$$

Put $V = \{z \in T: |z - 1| < \epsilon\}$, so that

(2) $$V_\epsilon^N = \{(z_j)_1^N \in T^N: \operatorname{Card}\{j: |z_j - 1| < \epsilon\} \geq (1 - \epsilon)N\}$$

for all $N \in \mathbf{N}$. We apply Lemma 1 to find an $N \in \mathbf{N}$ and an $F \in A(T^N)$ such that

(3) $$\operatorname{supp} F \subset V_\epsilon^N,$$

(4) $$\hat{F}(0) = 1 \quad \text{and} \quad |\hat{F}(k)| < \epsilon \quad \forall k \in \mathbf{Z}^N \setminus \{0\}.$$

Choose a finite set $L \subset \mathbf{Z}^N$ so that

$$\sum_{k \notin L} |\hat{F}(k)| \cdot C\|\tau\|_{PM} < \epsilon, \tag{5}$$

and put

$$\hat{K} = \bigcup_{k \in L} \{\chi \in \hat{G}: |(g^{(k)}\tau)\hat{}(\chi)| \geq \epsilon/\text{Card } L\}, \tag{6}$$

where $(k) = \Sigma_j k_j$ for $k = (k_1, \cdots, k_N) \in \mathbf{Z}^N$. Since $g^{(k)}\tau \in PF(G)$ and L is finite, $\hat{K}$ is compact. It follows from Lemma 2 that there are N characters $\chi_1, \cdots, \chi_N \in \hat{G}$ such that the sets

$$\chi_1^{-k_1} \cdots \chi_N^{-k_N} \hat{K} \quad (k \in L) \tag{7}$$

are pairwise disjoint. Note that the series

$$\sum_{k \in \mathbf{Z}^N} \hat{F}(k) g^{(k)} \overline{\chi}_1^{k_1} \cdots \overline{\chi}_N^{k_N}$$

converges to $F(g\overline{\chi}_1, \cdots, g\overline{\chi}_N) \in A(U)$ in the norm of $A(U)$ by (1). Setting

$$\tau' = F(g\overline{\chi}_1, \cdots, g\overline{\chi}_N)\tau = \sum_{k \in \mathbf{Z}^k} \hat{F}(k) g^{(k)} \overline{\chi}_1^{k_1} \cdots \overline{\chi}_N^{k_N} \tau, \tag{8}$$

we claim that τ' and $\chi_1, \cdots, \chi_N$ have the required properties if ϵ is replaced by $\epsilon/(C\|\tau\|_{PM} + 3)$.

To prove part (a), put

$$\nu = \sum_{0 \neq k \in L} \hat{F}(k) g^{(k)} \overline{\chi}_1^{k_1} \cdots \overline{\chi}_N^{k_N} \tau.$$

Then we have

$$|\hat{\nu}(\chi)| \leq \sum_{0 \neq k \in L} |\hat{F}(k)| \cdot |(g^{(k)}\tau)\hat{}(\chi_1^{k_1} \cdots \chi_N^{k_N} \chi)|$$

$$\leq \epsilon \sum_{k \in L} |(g^{(k)}\tau)\hat{}(\chi_1^{k_1} \cdots \chi_N^{k_N} \chi)|$$

for all $\chi \in \hat{G}$ by (4). If one of the summands in the last sum is $\geq \epsilon/\text{Card } L$, then the other summands are $< \epsilon/\text{Card } L$ by (6) and (7). Therefore, $|\hat{\nu}(\chi)| \leq \epsilon(C\|\tau\|_{PM} + 1)$ by (1); hence $\|\nu\|_{PM} \leq \epsilon(C\|\tau\|_{PM} + 1)$. It follows from (8), (4), and (5) that

$$\|\tau' - \tau\|_{PM} = \left\| \sum_{k \neq 0} \hat{F}(k) g^{(k)} \overline{\chi}_1^{k_1} \cdots \overline{\chi}_N^{k_N} \tau \right\|_{PM}$$

$$\leq \|\nu\|_{PM} + \sum_{k \notin L} |\hat{F}(k)| \cdot \|g^{(k)}\tau\|_{PM}$$

$$< \epsilon(C\|\tau\|_{PM} + 1) + \epsilon = \epsilon(C\|\tau\|_{PM} + 2).$$

Part (b) is trivial from (8).

To verify (c), note that supp τ' is contained in the closed set $\{x \in E$: Card$\{j$: $|g(x) - \chi_j(x)| \leq \epsilon\} \geq (1 - \epsilon)N\}$, which is easily seen from (2), (3), and (8). Thus $x \in$ supp τ' implies

$$\left| g(x) - N^{-1} \sum_{j=1}^{N} \chi_j(x) \right| \leq N^{-1} \sum_{j=1}^{N} |g(x) - \chi_j(x)| \leq \epsilon + N^{-1} \cdot 2\epsilon N = 3\epsilon.$$

This completes the proof.

Theorem 2. *Let G be a LCA I-group and τ a pseudo-function on G whose support E is totally disconnected and metrizable. Then for each $\epsilon > 0$ there exists a $\nu \in PF(E)$, with $\|\nu - \tau\|_{PM} < \epsilon$, whose support is a strongly independent H_1-set.*

Proof. Since $\hat{\tau}$ is uniformly continuous, we may assume that E is compact. Since E is metrizable, $C(E)$ is separable, and so $\{f \in C(E)$: $|f| = 1\}$ contains a countable dense set $\{g_n\}$. We may assume Card$[g_n(E)] < \infty$ for all n, because E is totally disconnected.

Let $\epsilon > 0$ be given, and set $\tau_0 = \tau$. Suppose that $\tau_0, \cdots, \tau_{n-1} \in PF(E)$ are constructed for some natural number n. We apply Lemma 3 to find a $\tau_n \in PF(E)$ which satisfies (a), (b), and (c) in Lemma 3 with τ, ϵ, g replaced by $\tau_{n-1}, \epsilon/2^n, g_n$, respectively. The sequence $\{\tau_n\}_1^\infty$ obtained in this way converges to some $\nu \in PF(E)$ and we have

$$\|\nu - \tau\|_{PM} \leq \sum_{n=1}^{\infty} \|\tau_n - \tau_{n-1}\|_{PM} < \epsilon.$$

Clearly supp $\nu \subset \bigcap_n$ supp τ_n. Thus, for each $n \in \mathbf{N}$, there are finitely many characters $\chi_1, \cdots, \chi_N \in \hat{G}$ such that

$$\left\| g_n - \frac{1}{N} \sum_{j=1}^{N} \chi_j \right\|_{C(\text{supp } \nu)} < \frac{\epsilon}{2^n}.$$

This implies that supp ν is an H_1-set which is independent over Z.

Lemma 2′. *Let G be a nondiscrete LCA group with $q = q(G) < \infty$, $E \subset G$ and $\hat{K} \subset \hat{G}$ compact, and $N \in \mathbf{N}$. Then there are N characters $\chi_1, \cdots, \chi_N \in \hat{G}$ with the following two properties:*

(i) *The sets $\chi_1^{k_1} \cdots \chi_N^{k_N} \hat{K}$ ($k_j \in \{0, 1, \cdots, q-1\}$, $1 \le j \le N$) are pairwise disjoint;*

(ii) *$\chi_j^q = 1$ on some neighborhood of E ($1 \le j \le N$).*

Proof. We may assume that G is compactly generated. Thus $G = R^a \times Z^b \times H$ for some nonnegative integers a and b and some compact abelian group H (see [1, (9.8)]). Since $q < \infty$, $a = 0$ and H is of bounded order. Let $\hat{K}_0$ be the natural projection of $\hat{K}$ into $\hat{H}$. Since H is a compact group with $q(H) = q(G) = q < \infty$, we can find N characters $\gamma_1, \cdots, \gamma_N \in \hat{H}$ with order q so that the sets $\gamma_1^{k_1} \cdots \gamma_N^{k_N} \hat{K}_0$ ($k_j \in \{0, 1, \cdots, q-1\}$, $1 \le j \le N$) are pairwise disjoint. (Note that $\hat{H}$ is a weak direct product of finite cyclic groups [4, B8].) Setting $\chi_j = 1 \otimes \gamma_j \in T^b \times \hat{H} = \hat{G}$, we see that the elements $\chi_1, \cdots, \chi_N$ have the required property.

Lemma 3′. *Let G be a nondiscrete LCA group with $q < \infty$, and τ a pseudo-function on G with compact support E. Let also $0 < \epsilon < 1$, and g any function in $C(E)$ with $g^q = 1$. Then there exists a $\tau' \in PF(G)$ such that*

(a) $\|\tau' - \tau\|_{PM} < \epsilon$;

(b) supp $\tau' \subset$ supp τ;

(c) $\|g - (1/N)\sum_{j=1}^N \chi_j\|_{C(\text{supp } \tau')} < \epsilon$

for some characters $\chi_1, \cdots, \chi_N \in \hat{G}$ with $\chi_j^q = 1$ on E.

The proof is very similar to that of Lemma 3. The needed modifications are as follows. Replace T and V by $T_q = \{z \in T: z^q = 1\}$ and $\{1\} \subset T_q$, respectively; realize the dual of T_q^N as $\{0, 1, \cdots, q-1\}^N$ in the usual way; put $L = \{0, 1, \cdots, q-1\}^N$; and use Lemma 2′ instead of Lemma 2.

The following theorem can be proved by applying Lemma 3′ just as Theorem 2 was proved. We omit the proof.

Theorem 3. *Let G be a nondiscrete LCA group with $q = q(G) < \infty$, and τ a pseudo-function on G whose support E is metrizable. Then, given $\epsilon > 0$, we can find a pseudo-function $\nu \in PF(E)$, with $\|\nu - \tau\|_{PM} < \epsilon$, whose support K has the following property: to each $\delta > 0$ and $g \in C(K)$ with $g^q = 1$, there correspond finitely many characters $\chi_1, \cdots, \chi_N \in \hat{G}$ such that $\chi_j^q = 1$ on K ($1 \le j \le N$) and $\|g - (1/N)\sum_{j=1}^N \chi_j\|_{C(K)} < \delta$.*

It is now a routine matter to derive Theorem 1 from Theorems 2 and 3 because every LCA group G contains a closed metrizable group H with $q(H) = q(G)$, and every Helson set of type M disobeys spectral synthesis (cf. [4, 5.6.10]).

Remarks. (a) For a characterization of Helson sets of (spectral) synthesis, we refer to [5].

(b) The totally disconnectedness assumption on E in Theorem 2 is unnecessary. Let E be a closed metrizable subset of G, and $\tau \in PF(E)$. Then there exists a sequence $\{\tau_n\}$ in $PF(E)$ such that supp τ_n is totally disconnected and $\|\tau_n - \tau\|_{PM} = o(1)$. We omit the proof.

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