HELSON SETS WHICH DISOBEY SPECTRAL SYNTHESIS

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ABSTRACT. In this paper it is shown that every nondiscrete LCA group contains a compact independent Helson set which disobeys spectral synthesis.

In [3] T. W. Körner has constructed an independent compact $H_1$-set of type $M$. This result solves negatively the long-standing problem whether or not every Helson set obeys spectral synthesis. Körner's construction of such a set is, however, very complicated, and R. Kaufman [2] has simplified it (see also [6]. In this paper, we modify Kaufman's method to prove that every nondiscrete metrizable LCA group contains an independent compact Helson set of type $M$. Consequently it is shown that every nondiscrete LCA group contains a Helson set which disobeys spectral synthesis.

Let $G$ be a LCA group with dual $\hat{G}$. We denote by $A(G)$ and $PM(G)$ the Fourier algebra on $G$ and the conjugate space of $A(G)$, respectively. Each element of $PM(G)$ is called a pseudo-measure on $G$. For $f \in A(G)$ and $P \in PM(G)$, we define

$$\langle f, P \rangle = (\hat{f} * \hat{P})(1) = \int_{\hat{G}} \hat{f}(\chi) \hat{P}(\chi^{-1}) d\chi,$$

where $\hat{f}$ and $\hat{P}$ denote the functions in $L^1(\hat{G})$ and in $L^\infty(\hat{G})$ whose (inverse) Fourier transforms are $f$ and $P$, respectively. If $\mu \in M(\hat{G})$ and $\tilde{\mu}(x) = \int_{G} \chi(x) d\mu(\chi)$ ($x \in G$), then we denote by $\tilde{\mu}P$ the pseudo-measure on $G$ defined by the requirement $(\tilde{\mu}P)\hat{\chi} = \mu \hat{\chi} \hat{P}$. It is well known that if $P \in PM(G)$ has compact support, then $\hat{P}$ can be chosen from the space $C(G)$; we will always do this. For such a $P$, $\langle \tilde{\mu}, P \rangle = (\mu \hat{P})(1) (\mu \in M(\hat{G}))$ is well defined, and we have $\langle \chi, \gamma P \rangle = \hat{P}(\gamma^{-1} \chi^{-1})$ ($\chi, \gamma \in \hat{G}$). A pseudo-function on $G$ is any pseudo-measure whose Fourier transform is a continuous function on $\hat{G}$ which vanishes at infinity. The space of all pseudo-functions is denoted by $PF(G)$.

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Let now $E$ be a closed set in $G$, and $q = q(G)$ the supremum of the natural numbers $n$ such that every neighborhood of $0_{G}$ contains an element with order $\geq n$. $E$ is called an $M$-set (or a set of type $M$) if $PF(E) \neq 0$ (that is, if $E$ carries a nonzero pseudo-function); strongly independent if $E$ is independent in the usual sense and $\text{ord}(e) = q$ for all $e \in E$; and an $H_{1}$-set if $\|\mu\|_{M} = \|\mu\|_{PM}$ for all $\mu \in M(E)$. In the case $q < \infty$, $E$ is called a $K_{q}$-set if every function $f \in C(E)$ with $f^{q} = 1$ is the restriction of a character in $\widehat{G}$; and a weak $K_{q}$-set if to each $\mu \in M(E)$ and $\epsilon > 0$ there corresponds a $K_{q}$-set $K \subset E$ such that $\mu(E \setminus K) < \epsilon$.

Our main result now follows (cf. [3, p. 105]).

Theorem 1. Every nondiscrete LCA group contains a strongly independent compact set which is either an $H_{1}$-set or a weak $K_{q}$-set but disobeys spectral synthesis. If, in addition, the group under consideration is metrizable, such a set can be chosen as a set of type $M$.

The proof becomes clear after some lemmas and theorems are established.

Lemma 1. To each $0 < \epsilon < 1$ there corresponds a natural number $N_{\epsilon}$ with the following property: for any neighborhood $V$ of $0$ of a compact abelian group $H$ and any natural number $N \geq N_{\epsilon}$, there exists an $F \in A(H^{N})$ such that

(i) $\text{supp } F \subset V_{\epsilon}^{N}$, and

(ii) $|\hat{F}(\hat{k})| < \epsilon |F(0)| \forall \hat{k} \in \hat{H}^{N} \setminus \{0\}$.

Here $V_{\epsilon}^{N}$ is the set of all $(x_{j})_{j=1}^{N} \in H^{N}$ such that $x_{j} \in V$ for at least $(1 - \epsilon)N$ indices $j = 1, 2, \cdots, N$, and $\hat{H}^{N}$ denotes the additively written dual of $H^{N}$.

The proof is essentially identical with that of [2, Lemma 2], so we omit it.

Lemma 2. Let $G$ be a LCA $I$-group, $\hat{K}$ a compact subset of $\hat{G}$, and $M, N \in N$ (the natural numbers). Then there exist $N$ characters $\chi_{1}, \cdots, \chi_{N} \in \hat{G}$ such that the sets $\chi_{1}^{k_{1}} \cdots \chi_{N}^{k_{N}} \hat{K}$ ($k_{j} \in \{0, \pm 1, \cdots, \pm M\}, 1 \leq j \leq N$) are pairwise disjoint.

Proof. By the structure theorem [1, (9.8)], $G$ contains an open subgroup $G_{0}$ which is topologically isomorphic to $R^{a} \times H$ for some $a \in \{0, 1, 2, \cdots\}$ and some compact abelian group $H$. Since there exists a continuous homomorphism of $\hat{G}$ onto $\hat{G}_{0}$, we may assume that $G = G_{0} = R^{a} \times H$.

Suppose that there exists a $y \in \hat{G}$ such that
Take $p \in \mathbb{N}$ so large that $n > p \quad \gamma^n \notin \hat{K}^{-1}\hat{K}$, and choose $N$ natural numbers $q_1, \ldots , q_N$ so that $|k_1q_1 + \cdots + k_Nq_N| > p$ for all choices of $k_j \in \{0, \pm 1, \pm 2, \ldots , \pm 2M\}$, $1 \leq j \leq N$, such that $(k_1, \ldots , k_N) \neq (0, \ldots , 0)$. Then the elements $\chi_j = y^{q_j}$, $1 \leq j \leq N$, have the required property.

Assume now that no $\gamma \in \hat{G}$ satisfies (1). Then $a = 0$, $G = H$ is compact and $\hat{G}$ is a torsion group. Therefore every finitely-generated subgroup of $\hat{G}$ is finite. But $\hat{G}$ is not of bounded order because $G$ is an I-group (see [4, 2.5.4]). Thus, setting $\chi_0 = 1$, we can find $\chi_1, \ldots , \chi_N \in \hat{G}$ so that

$$\text{ord} (\chi_j) > 2M \cdot \text{Card } [\hat{G}_p(\hat{K} \cup \{\chi_0, \ldots , \chi_{j-1}\})]$$

for all $j = 1, 2, \ldots , N$. As is easily seen, the $\chi_j$, $1 \leq j \leq N$, have the required property.

**Lemma 3.** Let $G$ be a LCA I-group, and $\tau$ a pseudo-function on $G$ whose support $E$ is compact. Let also $0 < \epsilon < 1$, and $g$ any function in $C(E)$ whose range is a finite subset of $T = \{z: |z| = 1\}$. Then there exist $\tau' \in PF(G)$ and $\chi_j \in \hat{G}$, $1 \leq j \leq N$, such that

(a) $\|\tau' - \tau\|_{PM} < \epsilon$,
(b) $\text{supp } \tau' \subseteq \text{supp } \tau$,
(c) $\|g - (1/N) \sum_{j=1}^{N} \chi_j\|_{C(\text{supp } \tau') < \epsilon}$.

**Proof.** We may assume that $g$ is defined and continuous on some compact neighborhood $U$ of $E$, and that $g(U)$ is a finite subset of $T$. Thus

(1) $C = \sup \{\|g^k\|_{A(U)}: k \in \mathbb{Z}\} < \infty,$

where

$$\|g^k\|_{A(U)} = \inf \{\|f\|_{A(G)}: f \in A(G), f = g^k \text{ on } U\}.$$ 

Put $V = \{z \in T: |z - 1| < \epsilon\}$, so that

(2) $V_\epsilon^N = \{(z_j)_1^N \in T^N: \text{Card } \{j: |z_j - 1| < \epsilon\} \geq (1 - \epsilon)N\}$

for all $N \in \mathbb{N}$. We apply Lemma 1 to find an $N \in \mathbb{N}$ and an $F \in A(T^N)$ such that

(3) $\text{supp } F \subseteq V_\epsilon^N$,

(4) $\hat{F}(0) = 1$ and $|\hat{F}(k)| < \epsilon$ for $k \in \mathbb{Z}^N \setminus \{0\}$.
Choose a finite set \( L \subseteq \mathbb{Z}^N \) so that
\[
\sum_{k \notin L} |\hat{F}(k)| \cdot C\|\tau\|_{PM} < \epsilon,
\]
and put
\[
\hat{K} = \bigcup_{k \in L} \{ \chi \in \hat{G}: |(g^{(k)}\tau)^\gamma(\chi)| \geq \epsilon/\text{Card } L \},
\]
where \((k) = \sum_j k_j \) for \( k = (k_1, \ldots, k_N) \in \mathbb{Z}^N \). Since \( g^{(k)}\tau \in PF(G) \) and \( L \) is finite, \( \hat{K} \) is compact. It follows from Lemma 2 that there are \( N \) characters \( \chi_1, \ldots, \chi_N \in \hat{G} \) such that the sets
\[
\chi_1^{-k_1} \cdots \chi_N^{-k_N} \hat{K} \quad (k \in L)
\]
are pairwise disjoint. Note that the series
\[
\sum_{k \in \mathbb{Z}^N} \hat{F}(k)g^{(k)}\chi_1^{k_1} \cdots \chi_N^{k_N}
\]
converges to \( F(g\chi_1, \ldots, g\chi_N) \in A(U) \) in the norm of \( A(U) \) by (1). Setting
\[
\tau' = F(g\chi_1, \ldots, g\chi_N)\tau = \sum_{k \in \mathbb{Z}^N} \hat{F}(k)g^{(k)}\chi_1^{k_1} \cdots \chi_N^{k_N}\tau,
\]
we claim that \( \tau' \) and \( \chi_1, \ldots, \chi_N \) have the required properties if \( \epsilon \) is replaced by \( \epsilon/(C\|\tau\|_{PM} + 3) \).

To prove part (a), put
\[
\nu = \sum_{0 \neq k \in L} \hat{F}(k)g^{(k)}\chi_1^{k_1} \cdots \chi_N^{k_N}\tau.
\]
Then we have
\[
|\hat{\nu}(\chi)| \leq \sum_{0 \neq k \in L} |\hat{F}(k)| \cdot |(g^{(k)}\tau)^\gamma(\chi_1^{k_1} \cdots \chi_N^{k_N}\chi)|
\leq \epsilon \sum_{k \in L} |(g^{(k)}\tau)^\gamma(\chi_1^{k_1} \cdots \chi_N^{k_N}\chi)|
\]
for all \( \chi \in \hat{G} \) by (4). If one of the summands in the last sum is \( \geq \epsilon/\text{Card } L \), then the other summands are \( < \epsilon/\text{Card } L \) by (6) and (7). Therefore, \( |\hat{\nu}(\chi)| \leq \epsilon(C\|\tau\|_{PM} + 1) \) by (1); hence \( \|\nu\|_{PM} \leq \epsilon(C\|\tau\|_{PM} + 1) \). It follows from (8), (4), and (5) that
Theorem 2. Let $G$ be a LCA $I$-group and $r$ a pseudo-function on $G$ whose support $E$ is totally disconnected and metrizable. Then for each $\epsilon > 0$ there exists a $\nu \in PF(E)$, with $\|\nu - r\|_{PM} < \epsilon$, whose support is a strongly independent $H_1$-set.

Proof. Since $\mathcal{F}$ is uniformly continuous, we may assume that $E$ is compact. Since $E$ is metrizable, $C(E)$ is separable, and so $\{f \in C(E): \|f\|_E = 1\}$ contains a countable dense set $\{g_n\}$. We may assume Card$[g_n(E)] < \infty$ for all $n$, because $E$ is totally disconnected.

Let $\epsilon > 0$ be given, and set $r_0 = r$. Suppose that $r_0, \cdots, r_{n-1} \in PF(E)$ are constructed for some natural number $n$. We apply Lemma 3 to find a $r_n \in PF(E)$ which satisfies (a), (b), and (c) in Lemma 3 with $r, \epsilon, g$ replaced by $r_{n-1}, \epsilon/2^n, g_n$, respectively. The sequence $\{r_n\}$ obtained in this way converges to some $\nu \in PF(E)$ and we have

$$\|\nu - r\|_{PM} \leq \sum_{n=1}^{\infty} \|r_n - r_{n-1}\|_{PM} < \epsilon.$$  

Clearly $\text{supp} \nu \subseteq \bigcap_n \text{supp} r_n$. Thus, for each $n \in \mathbb{N}$, there are finitely many characters $\chi_1, \cdots, \chi_N \in \hat{G}$ such that

$$\left\|g_n - \frac{1}{N} \sum_{j=1}^{N} \chi_j \right\|_{C(E)} < \frac{\epsilon}{2^n}.$$  

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This implies that \( \text{supp } \nu \) is an \( H_1 \)-set which is independent over \( \mathbb{Z} \).

**Lemma 2**. Let \( G \) be a nondiscrete LCA group with \( q = q(G) < \infty \), \( E \subset G \) and \( \hat{K} \subset \hat{G} \) compact, and \( N \in \mathbb{N} \). Then there are \( N \) characters \( \chi_1, \cdots, \chi_N \in \hat{G} \) with the following two properties:

(i) The sets \( \chi_1^k \cdots \chi_N^k \hat{K} \) (\( k_j \in \{0, 1, \cdots, q - 1\} \), \( 1 \leq j \leq N \)) are pairwise disjoint;

(ii) \( \chi_j^q = 1 \) on some neighborhood of \( E \) (\( 1 \leq j \leq N \)).

**Proof.** We may assume that \( G \) is compactly generated. Thus \( G = R^a \times Z^b \times H \) for some nonnegative integers \( a \) and \( b \) and some compact abelian group \( H \) (see [1, (9.8)]). Since \( q < \infty \), \( a = 0 \) and \( H \) is of bounded order. Let \( \hat{K}_0 \) be the natural projection of \( \hat{K} \) into \( \hat{H} \). Since \( H \) is a compact group with \( q(H) = q(G) = q \), we can find \( N \) characters \( \gamma_1, \cdots, \gamma_N \in \hat{H} \) with order \( q \) so that the sets \( \gamma_1^k \cdots \gamma_N^k \hat{K}_0 \) (\( k_j \in \{0, 1, \cdots, q - 1\} \), \( 1 \leq j \leq N \)) are pairwise disjoint. (Note that \( \hat{H} \) is a weak direct product of finite cyclic groups [4, B8].) Setting \( \chi_j = 1 \otimes \gamma_j \in T^b \times \hat{H} = \hat{G} \), we see that the elements \( \chi_1, \cdots, \chi_N \) have the required property.

**Lemma 3**. Let \( G \) be a nondiscrete LCA group with \( q < \infty \), and \( \tau \) a pseudo-function on \( G \) with compact support \( E \). Let also \( 0 < \epsilon < 1 \), and \( g \) any function in \( C(E) \) with \( g^q = 1 \). Then there exists a \( \tau' \in PF(G) \) such that

(a) \( \| \tau' - \tau \|_{PM} < \epsilon \);

(b) \( \text{supp } \tau' \subset \text{supp } \tau \);

(c) \( \| g - (1/N) \sum_{j=1}^N \chi_j \|_{C(\text{supp } \tau')} < \epsilon \)

for some characters \( \chi_1, \cdots, \chi_N \in \hat{G} \) with \( \chi_j^q = 1 \) on \( E \).

The proof is very similar to that of Lemma 3. The needed modifications are as follows. Replace \( T \) and \( V \) by \( T_{q} = \{ z \in T : z^q = 1 \} \) and \( \{1\} \subset T_q \), respectively; realize the dual of \( T_q^N \) as \( \{0, 1, \cdots, q - 1\}^N \) in the usual way; put \( L = \{0, 1, \cdots, q - 1\}^N \); and use Lemma 2 instead of Lemma 2.

The following theorem can be proved by applying Lemma 3 just as Theorem 2 was proved. We omit the proof.

**Theorem 3**. Let \( G \) be a nondiscrete LCA group with \( q = q(G) < \infty \), and \( \tau \) a pseudo-function on \( G \) whose support \( E \) is metrizable. Then, given \( \epsilon > 0 \), we can find a pseudo-function \( \nu \in PF(E) \), with \( \| \nu - \tau \|_{PM} < \epsilon \), whose support \( K \) has the following property: to each \( \delta > 0 \) and \( g \in C(K) \) with \( g^q = 1 \), there correspond finitely many characters \( \chi_1, \cdots, \chi_N \in \hat{G} \) such that

\( \chi_j^q = 1 \) on \( K (1 \leq j \leq N) \) and \( \| g - (1/N) \sum_{j=1}^N \chi_j \|_{C(K)} < \delta \).
It is now a routine matter to derive Theorem 1 from Theorems 2 and 3 because every LCA group $G$ contains a closed metrizable group $H$ with $q(H) = q(G)$, and every Helson set of type $M$ disobey spectral synthesis (cf. [4, 5.6.10]).

Remarks. (a) For a characterization of Helson sets of (spectral) synthesis, we refer to [5].

(b) The totally disconnectedness assumption on $E$ in Theorem 2 is unnecessary. Let $E$ be a closed metrizable subset of $G$, and $r \in \text{PF}(E)$. Then there exists a sequence $\{r_n\}$ in $\text{PF}(E)$ such that $\text{supp } r_n$ is totally disconnected and $\|r_n - r\|_{PM} = o(1)$. We omit the proof.

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