TRUNCATED POLYNOMIAL RINGS
OVER POINCARÉ ALGEBRAS

R. PAUL BEEM

ABSTRACT. There are given, in certain cases, necessary and suffi-
cient conditions for a truncated polynomial ring over a $\mathbb{Z}_2$-Poincaré alge-
bra to again be a Poincaré algebra. Applications are a splitting theorem
for Poincaré algebras and an algebraic bordism classification for real
projective space bundles.

1. Introduction. Let $H^*$ be a $\mathbb{Z}_2$-Poincaré algebra [1] and $x$ a class of
degree $m = 1, 2$. We will investigate the conditions under which the quotient
of the ring generated by $H^*$ and $x$ by the relation $x^n + a_1x + \cdots + a_n = 0$,
where $a_i$ is in $H^{im}$, is again a Poincaré algebra.

Among those spaces whose $\mathbb{Z}_2$-cohomology algebras form such rings are:

(i) the real (complex) projective space bundle of a real (complex) vector
bundle over a finite CW-complex. $a_i$ is the $i$th (2ith) Stiefel-Whitney class
of the vector bundle, and $x$ is the class of the $\mathbb{Z}_2(S^1)$ cover of the projective
space bundle by the sphere bundle of the vector bundles.

(ii) any $RP(2k)$ or $CP(2k)$ fibration over a smooth closed manifold.
(Such a fibration must be totally nonhomologous to zero, since $w_1$ pulls
back nontrivially.)

(iii) the Dold manifold $P(n, m) = S^n \times \mathbb{Z}_2 CP(m)$ [7].

(iv) any $P(1, 2k)$ fibration over a smooth closed manifold [2].

If $y = 1 + y_1 + \cdots + y_r$ satisfies the Wu relations for the action of $\bar{\Omega}(2)$,
the mod-2 Steenrod algebra, on a Stiefel-Whitney class, we will call $y$ an
"sw-class". See [4] for the Wu relations.

Our main results are

Theorem 1. If $m = 1$, then $K^* = H^*[x]/(x^n + a_1x^{n-1} + \cdots + a_n)$ is a
Poincaré algebra if and only if $a = 1 + a_1 + \cdots + a_n$ is an sw-class in $H^*$.
Theorem 2. If \( m = 2 \) and if \( H^*[x] = H^* \otimes \mathbb{Z}_2[x] \) is given the \( \mathbb{A}(2) \)-
structure induced by the relation \( Sq^1 x = bx \) for some \( b \) in \( H^1 \), then \( K^* \) is a
Poincaré algebra if \( (1 + b)^n + a_1(1 + b)^n - 1 + \cdots + a_n \) is an sw-class in \( H^* \).
Conversely, if \( K^* \) is a Poincaré algebra, where the degree of \( x \) is two and
\( Sq^1 x = bx \) for some \( b \) in \( H^1 \), then \( (1 + b)^n + a_1(1 + b)^n - 1 + \cdots + a_n \) is an
sw-class in \( H^* \).

As corollaries of Theorem 1, we show:

Theorem 3. If \( a = 1 + a_1 + \cdots + a_n \) is an sw-class in a Poincaré alge-
bra \( H^* \), then there is a Poincaré algebra \( L^* \) and a monomorphism of \( H^* \) into
\( L^* \) which takes \( a \) into a product \( \prod_{i=1}^n (1 + y_i) \), where \( y_i \) is in \( L^1 \) for every \( i \).

Theorem 4. If \( a = 1 + a_1 + \cdots + a_n \) is an sw-class in the Poincaré
algebra \( H^* \), then there is a vector bundle \( \eta \) over a smooth closed manifold \( M \)
such that (i) \( H^* \) and \( H^*(M; \mathbb{Z}_2) \) are (algebraically) bordant; and (ii) \( K^* \) and
\( H^*(\mathbb{P}(\eta); \mathbb{Z}_2) \) are (algebraically) bordant, where \( \mathbb{P}(\eta) \) denotes the total
space of the real projective bundle associated with \( \eta \). (One should compare
this with [5, Proposition 8.4].)

Theorem 5. \( w(K^*) = w(H^*)(\sum_{i=0}^n a_i(1 + x)^{n-i}) \), where \( w(H^*) \) and \( w(K^*) \)
are the "tangent" sw-classes constructed using duality and the \( \mathbb{A}(2) \)-
structure.

We will prove Theorems 1 and 2 in §2 and the corollaries in §3. All
algebras will be over \( \mathbb{Z}_2 \) and cohomology will be singular theory with \( \mathbb{Z}_2 \)
coefficients.

The author wishes to express his deep gratitude to his advisor, Pro-
fessor Robert E. Stong of the University of Virginia, for all his help and en-
couragement during the preparation of the author's dissertation of which this
paper is a portion.

2. Proof of Theorem 1. Let \( a = 1 + a_1 + \cdots + a_n \) be an sw-class in \( H^* \),
where \( a_i \) is in \( H^i \). Let
\[
\sigma: H^*(BO(n)) \to H^*(BO(n)) \otimes H^*(BO(1))
\]
be induced by the tensor product of the universal vector bundles over the
classifying spaces \( BO(n) \) and \( BO(1) \). Let \( \alpha: H^*(BO(n)) \to H^* \) take the un-
iversal sw-class \( 1 + w_1 + \cdots + w_n \) to \( a \), and

\[
\Psi: H^*(BO(n)) \otimes H^*(BO(1)) \to H^* \otimes H^*(BO(1))
\]
be $\alpha \otimes$ (identity). Finally, let
\[ \theta: H^* \otimes H^*(BO(1)) \rightarrow H^*[x] \]
be the natural identification. The composite $\theta(\Psi(a))$ takes $w_n$ to $x^n + a_1x^{n-1} + \cdots + a_n$ and is an $\tilde{A}(2)$-module homomorphism. Since $\text{Sq}^i w_n = w_{n+i}$ in $H^*(BO(n))$, $K^*$ admits an $\tilde{A}(2)$ action. Since the obvious homomorphism from $K^{n+p-1} \rightarrow \tilde{Z}_2$, where $\dim H^* = p$, does, in fact, provide duality for $K^*$, $K^*$ is a Poincaré algebra.

Conversely, suppose $K^*$ is a Poincaré algebra. Let $L^*$ denote the cohomology of the product $\prod_{i=1}^{\infty} K(Z, i)$, where $K(Z, i)$ is an Eilenberg-Mac Lane space. There is the homomorphism $\Psi \otimes$ (identity): $L^* \otimes \tilde{Z}_2[x] \rightarrow H^* \otimes \tilde{Z}_2[x]$, where $\Psi(a) = a_\tau$, $a_\tau$ being nonzero in $H^*(K(Z, r))$ and the quotient of $H^* \otimes \tilde{Z}_2[x]$ to $K^*$, both of which are $\tilde{A}(2)$-module homomorphisms. Denote the composite by $\phi: L^* \otimes \tilde{Z}_2[x] \rightarrow K^*$.

We claim that there are "universal" polynomials $q_{ij}(\tau_1, \cdots, \tau_n)$ in $L^*$ (i.e., independent of $H^*$ and $a$) such that $\phi(\text{Sq}^i \tau_j + q_{ij}) = 0$ for every $i$ and $j$. In fact, one may show that
\[
\sum_{j=1}^{n} x^{n-j} \phi(\text{Sq}^i \tau_j) = \text{Sq}^i \phi\left( \sum_{j=1}^{n} x^{n-j} \tau_j \right) + \sum_{l=1}^{n} \sum_{j=0}^{n-l} \binom{n-1}{l} x^{n-j+l} \phi(\text{Sq}^{i-l} \tau_j).
\]
But $\phi(\sum_{j=0}^{n} x^{n-j} \tau_j) = 0$, and inductively we can assume that $\phi(\text{Sq}^{i-l} \tau_j) = \phi(q_{i-l,j})$. Since there are universal polynomials $P_{rs}(\tau_1, \cdots, \tau_n)$ with
\[
x^{n-1+s} = \phi\left( \sum_{r=0}^{n-1} p_{rs} x^r \right),
\]
it follows that
\[
q_{ij}(\tau_1, \cdots, \tau_n) = \sum_{r=0}^{n-1} \sum_{s=1}^{i} \binom{n-r}{s} p_{n-j, s-r+1} q_{i-s, r}.
\]

Let $J$ be the ideal in $L^*$ generated by the elements $\text{Sq}^i \tau_j + q_{ij}$, $\beta: L^* \rightarrow L^*/J$ and $\beta'$ be the restriction of $\beta$ to the subring $\tilde{Z}_2[i_1, \cdots, i_n]$. It follows from the above paragraph that $\beta'$ is an epimorphism and that $J$ is contained in the kernel of $\Psi$.

Let $\alpha: L^* \rightarrow H^*(BO(n))$ be that usual homomorphism, the kernel of which is generated by the Wu relations. $J$ is contained in the kernel of $\alpha$ (which may be seen, for example, by taking for $H^*$ the cohomology of the product of $n$ copies of $RP(2n + 1)$). Since $Z_2[i_1, \cdots, i_n]$ is ring isomorphic
to $H^*(BO(n))$, $L^*/J$ is generated by the Wu relations. Therefore $\Psi$ annihilates the Wu relations and $1 + a_1 + \cdots + a_n$ is an sw-class. $\square$

Before proving Theorem 2, we need a preliminary result.

**Lemma 1.** If $H^*$ is a Poincaré algebra, $b$ in $H^1$, $g$ in $H^3$ and $Sq^1 g = bg$, then $H^*[x]$, where $x$ is given a degree of two, can be given a unique $\mathfrak{A}(2)$-structure by setting $Sq^1 x = bx + g$.

**Proof.** Recall that

$$H^*(K(Z_2, 2)) \cong \mathbb{Z}_2[\iota, Sq^1 \iota, Sq^2 \iota, \cdots, Sq^n \iota, \cdots],$$

where $Sq^m = Sq^2 Sq^2 \cdots Sq^1$. Let $L^* = H^* \otimes H^*(K(Z_2, 2))$ and consider the elements $u_i$, where $u_1 = Sq^1 \iota + b \iota + g$, and to get $u_{n+1}$ from $u_n$, one writes $u_n = Sq^{n-1} \iota + P_n(\iota)$, sets $Q_n(\iota, Sq^1 \iota) = Sq^2 P_n(\iota)$ and $P_{n+1}(\iota) = Q_n(\iota, b \iota + g)$ to get

$$u_{n+1} = Sq^n \iota + P_{n+1}(\iota).$$

One shows that

$$u_n = Sq^{n-1} \iota + \sum_{r=0}^{n-1} b(2^{n-2r+1} + 1) \iota 2^r + c_n,$$

where

$$c_1 = g \text{ and } c_{n+1} = Sq^n c_n + b(2^{n+1} - 2),$$

and then that

$$u_{n+1} = Sq^n u_n + b(2^{n+1} - 2)u_1.$$

If $J$ is the ideal in $L^*$ generated by the $u_i$'s, it follows that $J$ is an $\mathfrak{A}(2)$-ideal. Let $L^* = L^*/J$. Then $L^* \cong H^*[x]$ as rings, and we give $H^*[x]$ the $\mathfrak{A}(2)$-structure of $L^*$. $\square$

**Proof of Theorem 2.** Suppose $H^*[x]$ is given the $\mathfrak{A}(2)$-structure induced by $Sq^1 x = bx$ for some $b$ in $H^1$, and that $(1 + b)^n + a_1 (1 + b)^{n-1} + \cdots + a_n$ is an sw-class in $H^*$. Let $1 + w_1 + \cdots + w_{2n}$ be this class, where $w_i$ is in $H^i$. Then

$$L^* = H^*[y]/\langle y^{2n} + w_1 y^{2n-1} + \cdots + w_{2n} \rangle$$

is a Poincaré algebra, where the degree of $y$ is one. (One may think of $L^*$ as being the "RP-algebra" over the "CP-algebra", $K^*$.)

If $d = y^2 + yb$, 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
then $\text{Sq}^1 d = bd$ and $H^*[d]$ is, as an $\mathfrak{G}(2)$-submodule of $H^*[y]$, isomorphic to $H^*[x]$.

Let $\alpha: H^*[y] \rightarrow L^*$. Then

$$\alpha(H^*[d]) \cong H^*[1, d, \ldots, d^{n-1}],$$

as $H^*$-modules, and there will be some relation of the form $d^n + a_1 d^{n-1} + \cdots + a_n$. Therefore

$$\sum_{i=0}^{n} c_i (b + c)^i a_{n-i} = \sum_{j=0}^{2n} c_j w_{2n-j}.$$ 

But

$$\sum_{j=0}^{2n} c_j w_{2n-j} = \sum_{i=0}^{n} c_i (b + c)^i a_{n-i}.$$ 

Hence $a_i = a'_i$ and $\langle d^n + a_1 d^{n-1} + \cdots + a_n \rangle$ is an $\mathfrak{G}(2)$ ideal.

Conversely, suppose $K^*$ is a Poincaré algebra. Note that $K^*[y]/(y^2 + by + x)$ is also a Poincaré algebra, where the degree of $y$ is one. Clearly, this ring is freely generated over $H^*$ by $\{1, y, \ldots, y^n\}$, and there is a relation $\sum_{i=0}^{2n} y_i w_{2n-i} = 0$, for some $w_i$ in $H^i$. In fact,

$$1 + w_1 + \cdots + w_{2n} = (1 + b)^n + a_1 (1 + b)^{n-1} + \cdots + a_n.$$ 

This is an sw-class by Theorem 1. □

3. Before proceeding to Corollary 3, we need a preliminary result, which is probably well known.

**Lemma 2.** If $a = 1 + a_1 + \cdots + a_n$ and $b = 1 + b_1 + \cdots + b_m$ are sw-classes in an $\mathfrak{G}(2)$-algebra $X^*$, then so are $a^{-1}$ and $ab$.

**Proof.** There are the homomorphisms $\alpha$ and $\beta$ from $H^*(BO)$ to $X^*$ which send the universal sw-class $w$ to $a$ and $b$, respectively. Composing $\alpha$ with the $\mathfrak{G}(2)$-homomorphism $H^*(BO) \rightarrow H^*(BO)$, which inverts $w$, sends $w$ to $a^{-1}$. Hence $a^{-1}$ is an sw-class. Similarly, composing $\alpha \otimes \beta$ and multiplication in $X^*$ with the Whitney sum homomorphism $H^*(BO) \rightarrow H^*(BO) \otimes H^*(BO)$ sends $w$ to $ab$. □

**Proof of Theorem 3.** By Theorem 1, $K^*$ is a Poincaré algebra. Setting $b_i = \sum_{j=0}^{i} x^i a_{i-j}$ yields $a = (1 + x)(1 + b_1 + \cdots + b_{n-1})$. By the lemma, $b = 1 + b_1 + \cdots + b_{n-1}$ is an sw-class in $K^*$. The inclusion of $H^*$ in $K^*$ is a monomorphism. The result follows. □
Next, one recalls [3] the natural equivalence \( a^{N_*}(H^*(X)) \cong N_* (X) \) between the algebraic bordism of \( H^*(X) \) and the unoriented bordism of \( X \). If \( H^*(X) \to H^* \) and \( M \to X \), where \( H^* \) is a Poincaré algebra and \( M \) is a smooth manifold, correspond under this equivalence, we will call them "bordant." Similarly, we will say that an \( n \)-plane bundle \( \eta \) over \( M \) is bordant to \( H^*(BO(n)) \to H^* \) if the classifying map for \( \eta, f: M \to BO(n) \), is bordant to \( H^*(BO(n)) \to H^* \).

Let \( X^* \) be any left \( \mathfrak{A}(2) \)-algebra and suppose that \( f^*: X^* \otimes H^*(BO(n)) \to H^* \) is a left \( \mathfrak{A}(2) \)-algebra homomorphism, where \( H^* \) is a Poincaré algebra. Then there is a homomorphism

\[
X^* \otimes H^*(BO(n)) \otimes H^*(BO(1)) \to H^*[x]/(x^n + a_1x^{n-1} + \cdots + a_n),
\]

where the degree of \( x \) is 1, where \( \Sigma_i a_i \) is induced by the universal sw-class in \( H^*(BO(n)) \), which is \( f^* \) on \( X^* \otimes H^*(BO(n)) \) and which sends \( w_1 \) in \( H^1(BO(1)) \) to \( x \). The next result will imply Theorem 4.

**Lemma 3.** The above construction is well defined on algebraic bordism and yields a homomorphism

\[
a^{N_m}(X^* \otimes H^*(BO(n))) \to a^{N_m + m - 1}(X^* \otimes H^*(BO(n)) \otimes H^*(BO(1))).
\]

**Proof.** Suppose \( X^* \otimes H^*(BO(n)) \xrightarrow{f^*} H^* \) bounds, and that \( J^* \subset H^* \) denotes the self-annihilating subalgebra (with unity) closed under both left and right \( X^* \otimes H^*(BO(n)) \otimes \mathfrak{A}(2) \) action, the existence of which is equivalent to the hypothesis [6]. Since 1 is in \( J^* \), so is \( \Sigma a_i \). Let

\[
R^* = J^*[x]/(x^n + a_1x^{n-1} + \cdots + a_n) \subset K^*.
\]

We claim that \( R^* \) is a self-annihilating subalgebra closed under left and right action of \( X^* \otimes H^*(BO(n)) \otimes H^*(BO(1)) \otimes \mathfrak{A}(2) \), and therefore that \( K^* \) bounds. (Since the construction is clearly additive, this will prove the lemma.) But \( R^* \) is obviously closed under the left action. Since \( a^{-1} \) is a polynomial in \( a_i \), and is therefore in \( J^* \), one sees that \( R^* \) is contained in its annihilator, \( R' \). To show that \( R' \subset R^* \), let \( k_j = \Sigma_{i=0}^j b_i x^{j-i} \) be in \( (R')^n + m - 1 - j \) for \( b_i \) in \( H^i \). An induction on \( i \) shows that \( b_i \) is in \( J^{m-i} \), and therefore that \( k_j \) is in \( R^i \). Hence \( R' \subset R^* \). It follows that \( R^* \) is closed under the right action of \( X^* \otimes H^*(BO(n)) \otimes H^*(BO(1)) \otimes \mathfrak{A}(2) \). \( \square \)

Therefore, if \( a = \Sigma a_i \) in \( H^* \) is a given sw-class and \( f: M \to BO(n) \) is
bordant to it, $RP(f^*(y_n))$ with its canonical line bundle is bordant to $K^*$ with its $sw$-class $1 + x$. □

Proof of Theorem 5. Let $w(K^*) = 1 + w_1 + \cdots + w_{n+m-1}$ and $w(H^*)(\sum_{i=0}^n (1 + x)^i a_{n-1}) = 1 + u_1 + \cdots + u_{n+m-1}$. If $\tau: H^*(BO(m)) \to H^*$ is the "tangent" homomorphism, we have the homomorphism

$$\tau \otimes \alpha \otimes (id): H^*(BO(m)) \otimes H^*(BO(n)) \otimes H^*(BO(1))$$

$$\to H^* \otimes H^* \otimes \mathbb{Z}_2[x] \to K^*,$$

where $\alpha$ "realizes" $a$, and the second homomorphism is multiplication in $H^*$ followed by the projection to $K^*$. If $(\tau \otimes \alpha \otimes \text{id})(y_i) = u_i + w_i \neq 0$, there would be an $h$ in $H^S$ with $\mu_K(h(u_i + w_i) x^j) \neq 0$ and $h$ would define a homomorphism $\sigma: H^*(K(Z_2, s)) \to H^*$, by $\sigma(\iota) = h$, where $\iota$ is nonzero in $H^S(K(Z_2, s))$.

We would then have a manifold $M$ and a map $f: M \to BO(m) \times K(Z_2, s) \times BO(n)$ bordant to

$$\tau \otimes \sigma \otimes \alpha: H^*(BO(m)) \otimes H^*(K(Z_2, s)) \otimes H^*(BO(n)) \to H^*.$$

Setting $X = BO(m) \times K(Z_2, s)$, $X^* = H^*(X)$ and denoting the induced $n$-plane bundle over $M$ by $\eta$, we would get a map $RP(\eta) \to X \times BO(n) \times BO(1)$ bordant to $X^* \otimes H^*(BO(n)) \otimes H^*(BO(1)) \to K^*$.

But $y_i$ goes to zero in $H^*(RP(\eta))$ and hence, so does $\iota \otimes y_i \otimes w_i$. This contradiction establishes the result. □

REFERENCES