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ON THE PADÉ TABLE OF $\cos z$

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ABSTRACT. The Padé table for $\cos z$ and related functions is written in a form that involves the binomial coefficients, the Euler numbers and Bernoulli numbers. Identities involving the corresponding persymmetric determinants and their minors are developed.

1. **Introduction.** In the theory of the Padé table of $F(z) = c_0 + c_1 z + c_2 z^2 + \cdots$ appear the persymmetric determinants $\Delta_{m,n} = |c_{i-j+n}|$, $i, j = 1, \cdots, m+1$. Few tables are known in a closed form that do not involve these or closely related determinants. We find the Padé tables for $\cos z$, $(e^z - 1)/z$ and $(\sin z)/z$ in which the entries in the determinants are binomial coefficients, Euler numbers and Bernoulli numbers. The method employed enables us to rewrite persymmetric determinants and their minors as determinants of possibly much smaller dimension. A typical example is

$$\begin{vmatrix} 1/4! & 1/0! & 0 & & 0 \\ 1/6! & 1/2! & 1/0! & & 0 \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1/(2m+2)! & 1/(2m-2)! & 1/(2m-4)! & \cdots & 1/2! \end{vmatrix} = (-1)^{m+1} \begin{vmatrix} \dfrac{E_2}{2!} & \dfrac{E_0}{0!} \\ \dfrac{E_{2m+2}}{(2m+2)!} & \dfrac{E_{2m}}{(2m)!} \end{vmatrix}.$$

Such rewritings will in some instances shorten the computation considerably.

2. The Padé table [1] of a power series $F(z) = c_0 + c_1 z + c_2 z^2 + \cdots$, $c_0 \neq 0$, is the table of unique rational functions $[m, n, F] = [m, n] = P_{m,n}(z)/Q_{m,n}(z)$ satisfying

(1) for some integer $c \geq 0$ the degree of $x^c P_{m,n}$ is $\leq n$, the degree of $x^c Q_{m,n}$ is $\leq m$, $P_{m,n}(0) = c_0$, $Q_{m,n}(0) = 1$, $P_{m,n}/Q_{m,n}$ is in lowest terms, and

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(2) $x^c(Q_{m,n}F - P_{m,n}) = (z^{m+n+1})$, where (z^a) denotes any power series where all terms of degree $< a$ are missing.

If we set

$$x^c P_{m,n} = a_0 + a_1 z + \cdots + a_n z^n \quad \text{and} \quad x^c Q_{m,n} = b_0 + b_1 z + \cdots + b_m z^m,$$

then (2) leads to the two systems of equations

$$\sum_{k=0}^{m} c_{s-k} b_k = \begin{cases} a_s & s = 0, 1, \cdots, n, \\ & \text{for} \\ 0 & s = n+1, \cdots, n+m, \end{cases} \tag{3}$$

where we set $c_i = 0$ when $i < 0$. To solve these systems of equations we introduce some determinants:

$$\Delta_{m,n} = \begin{vmatrix} c_n & c_{n-1} & \cdots & c_{n-m} \\ c_{n+1} & c_n & \cdots & c_{n-m+1} \\ \cdot & \cdot & \cdot & \cdot \\ c_{n+m} & c_{n+m-1} & \cdots & c_n \end{vmatrix}, \qquad m, n = 0, 1, 2, \cdots.$$

This can also be written $\Delta_{m,n} = |c_{i-j+n}|$, $i, j = 1, 2, \cdots, m+1$. Also $\Delta_{-1,n} = 1$, $n = 0, 1, \cdots$; $\Delta_{m,n}(z)$ is obtained from $\Delta_{m,n}$ when the first row is replaced by $1, z, \cdots, z^m$; $\Delta_{m,n}(c_k)$ is obtained from $\Delta_{m,n}$ when the first row is replaced by $c_k, c_{k-1}, \cdots, c_{k-m}$, and we have therefore

$$\Delta_{m,n}(c_k) = |c_{i-j+n}|, \qquad i = k+1-n, 2, 3, \cdots, m+1;\ j = 1, 2, \cdots, m+1,$$

and finally $\overline{\Delta_{m,n}(c_{n-k})}$ is the minor of $\Delta_{m,n}$ which belongs to c_{n-k} in the first row of $\Delta_{m,n}$.

Assuming $\Delta_{m-1,n} \neq 0$, we may solve equations (3) and find

$$P_{m,n} = \sum_{k=0}^{n} \frac{\Delta_{m,n}(c_k) z^k}{\Delta_{m-1,n}}, \qquad Q_{m,n} = \frac{\Delta_{m,n}(z)}{\Delta_{m-1,n}}. \tag{4}$$

Few Padé tables are known in a form not involving the above determinants. If we set $1/F(z) = d_0 + d_1 z + d_2 z^2 + \cdots$, we have $d_m = (-1)^m c_0^{-m-1} \Delta_{m-1,1}$. But an easier way to determine d_m is recursively by $d_0 = c_0^{-1}$, $d_m c_0 + d_{m-1} c_1 + \cdots + d_0 c_m = 0$, $m = 1, 2, \cdots$. The determinants analogous to the Δ's will be denoted by $D_{m,n}$, etc. when the c's are replaced by the d's.

Equation (2) implies that

$$x^c(P_{m,n}F^{-1} - Q_{m,n}) = -(z^{m+n+1})F^{-1} = (z^{m+n+1}),$$

which shows that

$$[n, m, 1/F] = 1/[m, n, F] = c_0^{-1}Q_{m,n}/c_0^{-1}P_{m,n}, \tag{5}$$

where the factor c_0^{-1} normalizes the fraction as required in (1). Formulas (4) thus yield

$$c_0^{-1}Q_{m,n} = \frac{c_0^{-1}\Delta_{m,n}(z)}{\Delta_{m-1,n}} = \sum_{k=0}^{m} \frac{D_{n,m}(d_k)z^k}{D_{n-1,m}}, \tag{6}$$

where we assume $D_{n-1,m} \neq 0$. Comparing coefficients we find

$$(-1)^k\Delta_{m,n}\overline{(c_{n-k})}/\Delta_{m-1,n} = c_0D_{n,m}(d_k)/D_{n-1,m}, \tag{7}$$

and in particular for $k = m$,

$$(-1)^m\Delta_{m-1,n+1}/\Delta_{m-1,n} = c_0D_{n,m}/D_{n-1,m}.$$

If we replace n by $0, 1, \cdots, n-1$ and multiply the n equations, we obtain

$$(-1)^{mn}\Delta_{m-1,n}/\Delta_{m-1,0} = c_0^nD_{n-1,m}/D_{-1,m},$$

and using $\Delta_{m-1,0} = c_0^m$, $D_{-1,m} = 1$, we find

$$\Delta_{m-1,n} = (-1)^{mn}c_0^{m+n}D_{n-1,m}. \tag{8}$$

Most of the above development appears in [2]. Substituting (8) into (7) yields

$$\Delta_{m,n}\overline{(c_{n-k})} = (-1)^{mn}c_0^{m+n+1}D_{n,m}(d_k). \tag{9}$$

Also, (8) shows that the assumption $D_{n-1,m} \neq 0$ following (6) is unnecessary since (8) clearly is a polynomial identity and $\Delta_{m-1,n}$ is assumed $\neq 0$.

3. We turn to $F(z) = \cos z = \Sigma_0^\infty (-1)^m z^{2m}/(2m)!$, which is an even function. Set $z^2 = w$ and seek the Padé table for

$$\cos\sqrt{w} = \sum_0^\infty \frac{(-1)^m w^m}{(2m)!}.$$

From

$$Q_{m,n}(w)\cos\sqrt{w} - P_{m,n}(w) = (w^{m+n+1})$$

follows

$$Q_{m,n}(z^2)\cos z - P_{m,n}(z^2) = (z^{2m+2n+2})$$

where $Q_{m,n}(z^2)$ is of degree $\leq 2m$, $P_{m,n}(z^2)$ of degree $\leq 2n$, both being even polynomials. Thus

$$\frac{P_{m,n}(z^2)}{Q_{m,n}(z^2)} = [2m,\ 2n,\ \cos z] = [2m+1, 2n] = [2m, 2n+1] = [2m+1, 2n+1].$$

The Padé table for $\cos z$ (or any even function) consists therefore of square blocks of dimension $2r \times 2r$. We believe that for $\cos z$ all blocks are of size 2×2 but are unable to prove this. We need a lemma.

Lemma. *The determinants* $|a_{ij}|$ *and* $|(-1)^{i+j}a_{ij}|$, $i, j = 1, \cdots, m$, *are equal.*

The proof is almost trivial.

Theorem 1. *For* $F(z) = \cos z$, *the Padé approximants are given by*

$$P_{2m,2n} = \sum_{k=0}^{n} \frac{(-1)^k z^{2k}}{(2k)!K_{2m-2,2n}} \left| \binom{2n+2i-2}{2j-2} \right|_{\substack{i=k+1-n,2,\cdots,m+1,\\ j=1,2,\cdots,m+1}},$$

$$Q_{2m,2n} = \frac{1}{K_{2m-2,2n}} \left| \begin{matrix} 1 & -z^2/2! & \cdots & (-1)^m z^{2m}/(2m)! \\ & \multicolumn{3}{c}{\binom{2n+2i-2}{2j-2}} \end{matrix} \right|_{\substack{i=2,\cdots,m+1,\\ j=1,2,\cdots,m+1}},$$

where

$$K_{2m-2,2n} = \binom{2n+2m}{2n} \left| \binom{2n+2i-2}{2j-2} \right|_{i,j=1,2,\cdots,m}$$

is assumed $\neq 0$.

Proof. In the determinant formulas (4) for $P_{m,n}(\sqrt{w})$ and $Q_{m,n}(\sqrt{w})$, we remove the checkerboard minus signs from the coefficients $(-1)^m/(2m)!$ using the Lemma. If we then in Q multiply the 2nd, 3rd, $\cdots$, $(m+1)$st row of the numerator with $(2n+2)!$, $(2n+4)!$, $\cdots$, $(2n+2m)!$, respectively, and columns

$2, 3, \cdots, m+1$ by $1/2!, 1/4!, \cdots, 1/(2m)!$, respectively, and in the denominator multiply rows $1, 2, \cdots, m$ by $(2n)!, (2n+2)!, \cdots, (2n+2m-2)!$ and columns $2, 3, \cdots, m$ by $1/2!, 1/4!, \cdots, 1/(2m-2)!$, respectively, we arrive at the above expression for Q. A similar computation yields P. We note that when $i < 0$ so that $c_i = 0$ the corresponding binomial coefficients are indeed zero.

The fact that the above formula for $[m, n, \cos z]$ involves determinants whose entries are integers (binomial coefficients) enables us to compute it with greater accuracy and speed.

No closed form for $K_{2m,2n}$ seems to be known. The above formulas are useful in computing the first few rows of the cosine table since the determinants involved are of dimension m and $m+1$ for both of the (identical) rows, numbered $2m$ and $2m+1$. To compute the columns we use the following theorem.

Theorem 2. *For $F(z) = \cos z$ the Padé approximants are given by*

$$P_{2m,2n} = \frac{1}{L_{2m,2n-2}} \begin{vmatrix} 1 & \cdots & (-1)^n z^{2n}/(2n)! \\ \binom{2m+2i-2}{2j-2} E_{2m+2i-2j} & & \end{vmatrix}_{\substack{i=2,\cdots,n+1,\\ j=1,\cdots,n+1}},$$

$$Q_{2m,2n} = \sum_{l=0}^{m} \frac{(-1)^l z^{2l}}{(2l)! L_{2m,2n-2}} \left| \binom{2m+2i-2}{2j-2} E_{2m+2i-2j} \right|_{\substack{i=l+1-m,2,\cdots,n+1,\\ j=1,2,\cdots,n+1}}$$

where

$$L_{2m,2n-2} = \binom{2m+2}{2n} \left| \binom{2m+2i-2}{2j-2} E_{2m+2i-2j} \right|_{i,j=1,2,\cdots,n}$$

is assumed $\neq 0$. We set $E_i = 0$ when $i < 0$ while $E_0, E_2, \cdots$ are the Euler numbers recursively defined by

$$E_0 = 1, \qquad \sum_{j=0}^{2m} \binom{2m}{2j} E_{2j} = 0, \qquad m = 1, 2, \cdots.$$

The proof follows the same lines as that of Theorem 1 when we use $[m, n, \cos z] = 1/[n, m, \sec z]$ and that $\sec z = \sum_{k=0}^{\infty} (-1)^k E_{2k} z^{2k}/(2k)!$. We note that the dimensions of the determinants are n and $n+1$ in the identical columns $2n$ and $2n+1$.

4. Next we consider $(e^z - 1)/z = \Sigma_0^\infty z^n/(n+1)!$ and recall that

$$\frac{z}{e^z - 1} = \sum_0^\infty \frac{B_m z^m}{m!},$$

where the B_m's are the Bernoulli numbers determined recursively by $B_0 = 1$, $B_1 = -\frac{1}{2}$, $B_3 = B_5 = \cdots = 0$ and $\Sigma_{j=0}^n \binom{n+1}{j} B_j = 0$. If $m \leq n+1$ and $[m, n, e^z] = \hat{P}_{m,n}/\hat{Q}_{m,n}$, then the polynomial $(\hat{P}_{m,n+1} - \hat{Q}_{m,n+1})/z$ is of degree $\leq n$ and

$$\hat{Q}_{m,n+1}(e^z - 1)/z - (\hat{P}_{m,n+1} - \hat{Q}_{m,n+1})/z = (z^{m+n+1}),$$

which shows that

$$[m, n, (e^z - 1)/z] = (\hat{P}_{m,n+1} - \hat{Q}_{m,n+1})z^{-1}/\hat{Q}_{m,n+1}.$$

The remainder of the table, $m \geq n+2$ (even $m \geq n+1$) may be computed as follows.

By (5) and (4) we have $P_{m,n} = c_0^{-1} D_{n,m}(z)/D_{n-1,m}$, and since $m \geq n+1$, neither of the determinants contains d_1. Thus $P_{m,n}$ might as well have been generated by $\Sigma_0^\infty B_{2m} z^{2m}/(2m)!$, an even function. As in the case of $\cos z$, we conclude

$$P_{2m,2n} = P_{2m+1,2n} = P_{2m,2n+1} = P_{2m+1,2n+1}$$

and similar equations for the Q's since they are uniquely determined by $1/F$ and P. The Q's, however, are not even since $1/F$ is not, but are determined by

$$Q_{2m,2n} = b_0 + b_1 z + \cdots + b_{2m} z^{2m},$$

$$\sum_{k=0}^{2n} \frac{B_k a_{s-k}}{k!} = b_s, \qquad s = 0, 1, \cdots, 2m,$$

where the a's are the coefficients of

$$P_{2m,2n} = \frac{1}{M_{2m,2n-2}} \begin{vmatrix} 1 & \cdots & z^{2n}/(2n)! \\ \binom{2m+2i-2}{2j-2} B_{2m+2i-2j} & & \end{vmatrix}_{\substack{i=2,3,\cdots,n+1, \\ j=1,2,\cdots,n+1}},$$

and

$$M_{2m,2n-2} = \binom{2m+2n}{2n} \left| \binom{2m+2i-2}{2j-2} B_{2m+2i-2j} \right|_{i,j=1,2,\cdots,n}$$

is assumed different from zero.

For $z^{-1}\sin z$ the same technique yields

$$P_{2m,2n} = \sum_{k=0}^{n} \frac{(-1)^k z^{2k}}{(2k+1)!N_{2m-2,2n}} \left| \binom{2n+2i-1}{2j-2} \right|_{\substack{i=k+1-n,2,\cdots,m+1,\\ j=1,2,\cdots,m+1}},$$

$$Q_{2m,2n} = \frac{1}{N_{2m-2,2n}} \left| \begin{matrix} 1 & -z^2/2! & \cdots & (-1)^m z^{2m}/(2m)! \\ & \binom{2n+2i-1}{2j-2} & & \end{matrix} \right|_{\substack{i=2,3,\cdots,m+1,\\ j=1,2,\cdots,m+1}},$$

where

$$N_{2m-2,2n} = \binom{2n+2m+1}{2m} \left| \binom{2n+2i-1}{2j-2} \right|_{i,j=1,2,\cdots,m}$$

is assumed different from zero.

5. We finally note that (9) enables us to express certain minors of persymmetric determinants by means of other determinants of possibly much smaller order, thus considerably facilitating computation. As examples, we find by the above technique applied to cos z that

$$\left| \frac{1}{(2n+2i-2j+2)!} \right|_{\substack{i=1,\cdots,m,\\ j=1,\cdots,m+1,j\neq k+1}} = (-1)^{mn+k} \left| \frac{E_{2m+2i-2j}}{(2m+2i-2j)!} \right|_{\substack{i=k+1-m,2,\cdots,n+1,\\ j=1,2,\cdots,n+1}},$$

and, in particular for $n = k = 1$, we obtain the example in the introduction.

From the expansion of $(e^z - 1)/z$ follows

$$\left| \frac{1}{(n+i-j+2)!} \right|_{\substack{i=1,\cdots,m,\\ j=1,\cdots,m+1,j\neq k+1}} = (-1)^{mn+k} \left| \frac{B_{m+i-j}}{(m+i-j)!} \right|_{\substack{i=k+1-m,2,\cdots,n+1,\\ j=1,\cdots,n+1.}}.$$

LITERATURE

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