TRANSFORMATION GROUPS RESEMBLING THE ADJOINT REPRESENTATION

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ABSTRACT. If $G$ is a compact, connected Lie group, the isotropy subgroups of the adjoint representation of $G$ are connected and the dimension of the fixed point set of a maximal torus of $G$ is equal to the rank of $G$. Results similar to these are given when $G$ acts differentiably on an integral cohomology sphere and has the adjoint representation as weak linear model. This is done by analyzing an induced action of the Weyl group of $G$.

1. Let $G$ be a compact, connected Lie group. $G$ acts on itself by conjugation, fixing the identity element and thereby giving rise to an orthogonal representation of $G$ on the tangent space at the identity. This representation, called the adjoint representation, is completely described by its restriction to a maximal torus $T$ of $G$, and this restriction is in turn completely described by the roots of $G$ which are considered as vectors in the Lie algebra of $T$. The Lie algebra of $T$ is contained in the Lie algebra of $G$; since it is fixed by $T$ it is acted on by $NT/T = W$, where $NT$ is the normalizer of $T$ in $G$. The finite group $W$, called the Weyl group of $G$, permutes the roots of $G$ so that if $\alpha$ is a root, so is $\sigma \alpha$ for any $\sigma$ in $W$. $W$ is generated by the orthogonal reflections through the hyperplanes perpendicular to the (non-zero) roots. If $S_\alpha$ is the reflection through the plane perpendicular to $\alpha$ and $\sigma$ is an element of $W$, $\sigma S_\alpha \sigma^{-1}$ is reflection through the root $\sigma \alpha$. Every reflection in $W$ is of the form $S_\beta$ for some nonzero root $\beta$ [2, p. 74]. $W(G)$ will also be used to denote the Weyl group of $G$.

Let $F(T)$ denote the fixed point set of $T$; $G^0_x$ the subgroup of elements of $G$ which fix $x$ (called the isotropy subgroup); $G^0_x$ the connected component of $G_x$ which contains the identity. If $g$ is an element of $G$ which fixes $x$ but does not lie in $G^0_x$, then $g^{-1}Tg = k^{-1}Tk$ for some $k$ in $G^0_x$ since all maximal tori of $G^0_x$ are conjugate, so $gk^{-1}$ is an element of $W(G)_x$ when $x$ is fixed by $T$. If $\alpha$ is a nonzero root and $H_\alpha$ is the plane fixed by $S_\alpha$, the centralizer in $G$ of the corank one subtorus of $T$ corresponding to $H_\alpha$
is denoted by \( G_\alpha \). Since \( G_\alpha \) is connected [3, pp. 26—31] and \( W(G_\alpha) \) is generated by \( S_\alpha \), any point in \( F(T) \) fixed by \( S_\alpha \) is also fixed by \( G_\alpha \). Since \( W(G)_x \) is generated by reflections which fix \( x \), we have

**Proposition 1.**
(a) If \( x \in F(T) \) is fixed by \( S_\alpha \), \( G_\alpha \subset G_x \).
(b) The isotropy subgroups of the adjoint representation are connected.
(c) For \( x \in F(T), W(G)_x = W(G)_x \).

2. We want to generalize these observations to differentiable actions on cohomology spheres. If \( M \) is a differentiable manifold, a diffeomorphism reflection on \( M \) is a diffeomorphism \( r \) such that (a) \( r \) reverses orientation and \( r^2 = 1 \); (b) the complement of the fixed point set of \( r \) is disconnected. We shall refer to diffeomorphism reflections simply as reflections. The fixed point set of a reflection on an integral cohomology sphere \( HS^n \) is a \( Z_2 \) cohomology sphere by Smith theory. However, a standard argument [6, p. 35] shows that in fact it is an integral cohomology sphere \( HS^{n-1} \).

If \( W \) is a group which is the Weyl group of a semisimple Lie group \( G \), the rank of \( W \) is defined as the rank of \( G \) (i.e. the dimension of a maximal torus of \( G \)). For such a group \( W \) we consider differentiable actions \( (W, \phi, HS^k) \). Such an action is called effective if no element of \( W \) fixes every element of \( HS^k \). Such an action is called normal if (a) each linear reflection in \( W \) is a reflection on \( HS^k \); (b) no product of two distinct linear reflections in \( W \) acts trivially. Considering \( \phi \) as a homomorphism of \( W \) into the group of diffeomorphisms of \( HS^k \), we say that a reflection on \( HS^k \) is proper if it is the image under \( \phi \) of a linear reflection in \( W \).

**Theorem 2.** If \( (W, \phi, HS^k) \), \( k \geq 1 \) is a normal action, then it is effective and each reflection is proper.

**Proof.** The proof is by induction on the rank of \( W \). If rank \( W = 1 \), the theorem follows trivially from the definition. Let rank \( W = n > 1 \) and suppose that the theorem is true for rank \( W \) less than \( n \). If \( S \) is a proper reflection there is a point \( x \) in \( HS^k \) which is fixed by \( S \) but which is not fixed by any other proper reflection since the proper reflections have distinct fixed point sets (which meet transversally). If \( t \) is any other element of \( W \) fixing such an \( x \), then \( t^{-1}St = S \). Hence any element of \( W \) acting trivially must lie in the center of \( W \). If \( w \) is such an element, the eigenvalues of \( w \) are \( \pm 1 \); but if \( w \) takes \( +1 \) as an eigenvalue, then \( w \) lies in a Weyl subgroup of \( W \) of less rank by Proposition 1(c). So we may assume that \( w = -1 \).

**Lemma.** The element \(-1\) in \( W \) can be written as the product of \( n \) mutually commuting distinct linear reflections where \( n = \text{rank} \ W \).
Proof of Lemma. It is sufficient to prove this when \( W \) is the Weyl group of a simple Lie group. If \( W = W(E_6), W(A_q), q \neq 1, \) or \( W(D_p), p \) odd, then \(-1\) is not an element of \( W \) [2, p. 284]. For the remaining classical groups it is obvious, and since \( W(B_4) \) is contained in \( W(F_4) \) and \( W(D_8) \) is contained in \( W(E_8), \) we are left with \( G_2 \) and \( E_7. \) We use the notation of [2, pp. 250–275]. For \( G_2 \) reflect through roots in the set \( \{\epsilon_1, \epsilon_2\}. \) For \( E_7 \) use roots in the set \( \{\epsilon_1 - \epsilon_2, \epsilon_1 + \epsilon_2, \epsilon_3 - \epsilon_4, \epsilon_3 + \epsilon_4, \epsilon_5 - \epsilon_6, \epsilon_5 + \epsilon_6, \epsilon_7 - \epsilon_8\}. \)

Returning to the proof of the theorem, we have \( w = R_1 R_2 \cdots R_n \) acting trivially, \( R_i R_j = R_j R_i. \) Since each \( R_i \) reverses orientation, \( n \) must be even. But each \( R_i \) acts on the fixed point set of \( R_n, F(R_n), \) and must reverse its orientation (for \( i \neq n) \) since it does so locally near a fixed point. Hence the case \( n \) even is also impossible.

If \( s \) in \( W \) is an improper reflection we have essentially the same situation. Since \( s^2 = 1, \) either \( s \) lies in a Weyl subgroup of smaller rank or \( s = -1, \) so we may write \( s = R_1 R_2 \cdots R_n \) in accordance with the Lemma above. Each \( R_i \) acts on \( F(s) \) and reverses the orientation (or interchanges the two fixed points when \( k = 1). \) Since the product acts trivially on \( F(s) \) we are done.

3. Let \((G, \phi, HS^n)\) be a differentiable action of a compact connected Lie group \( G \) on an integral cohomology sphere \( HS^n. \) Let \( S(\phi) \) be the \( CW \) system of \( \phi \) [8]. \( S(\phi) \) is defined as a collection of linear functionals on the Lie algebra of \( T \) which vanish on the Lie algebras of the connected co-rank one subtori of \( T \) which have fixed point sets of greater dimension than that of \( T. \) \( S(\phi) \) includes the zero functional with multiplicity \( dF(T) + 1, \) where \( dF(T) \) denotes the dimension of the fixed point set of \( T; S'(\phi) \) denotes the collection of nonzero elements in \( S(\phi). \) We assume for the remainder of this section that \( S'(\phi) = S'(Ad_G); \) the elements of this set then vanish on the hyperplanes perpendicular to the roots of \( G \) discussed in \( \S 1. \) The fixed point set of \( T \) is an integral cohomology sphere acted upon by \( W(G). \) The fact that this is a normal action is proved in [8, 1.11] and will not be repeated here. It is an easy consequence of a theorem of W.-Y. Hsiang which states that Proposition 1(a) holds for differentiable actions (cf. [5, Proposition 3, p. 349]).

Since \( W(G)_x \) is generated by the reflections which fix \( x \) [4, Theorem 1] and since these are now known to be proper reflections, it follows, just as in the proof of Proposition 1, that the isotropy subgroups are connected and that \( W(G)_x = W(G_x) \) for \( x \) in \( F(T). \) That \( T \) is a principal isotropy subgroup of \( \phi \) now follows from [2, Theorem 3.3]. Since the \( W(G) \) action is now
known to be effective, it follows from Theorem 3 of [4] that \( dF(T) \geq \text{rank } G - 1 \). Thus we have

**Theorem 3.** Let \((G, \phi, HS^n)\) be a differentiable action of a compact connected Lie group with \( S'(\phi) = S' (\text{Ad}_G) \). Then:

(a) \( dF(T) \geq \text{rank } G - 1 \);

(b) All isotropy subgroups are connected;

(c) For \( x \) in \( F(T) \), \( W(G)_x = W(G_x) \).

(d) \( S(\phi) = S(\text{Ad}_G + r \text{ trivial copies}) \), where \( r = dF(T) - \text{rank } G + 1 = dF(G) + 1 \).

**Remark.** Part of the proof of Theorem 3 of [4] has not yet appeared, namely the proof of Theorem 2'. In lieu of this, one can produce a weaker inequality in (a) above as follows: \( W(G) \) contains subgroups isomorphic to \( Z_2 \times \cdots \times Z_2 = (Z_2)^k \), where each copy of \( Z_2 \) is generated by a reflection in \( W(G) \). If \(-1\) is an element of \( W(G) \), the Lemma shows that \( k = \text{rank } G \); otherwise \( k \) is at least \( \frac{1}{2} \text{rank } G \). One can apply the formula of Borel [1, Chapter XIII, Theorem 2.3] with \( Z_2 \) coefficients to these subgroups to show that \( dF(T) \) is at least \( \frac{1}{2} \text{rank } G \).

**REFERENCES**


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