A REFORMULATION OF THE RADON-NIKODYM THEOREM
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ABSTRACT. The Radon-Nikodym theorems of Segal and Zaanen are principally concerned with the classification of those measures $\mu$ for which any $\lambda \ll \mu$ is given in the form

(i) $\lambda(A) = \int_A g \, d\mu$

for all sets $A$ of finite $\mu$ measure.

This paper is concerned with the characterization of those pairs $\lambda, \mu$ for which the equality (i) holds for every measurable set $A$, and introduces a notion of compatibility that essentially solves this problem. In addition, some applications are made to Radon-Nikodym theorems for regular Borel measures.

1. Introduction. The sharpest known form of the Radon-Nikodym theorem to date is due to Segal and appears in Segal [6] and Zaanen [7]. The purpose of this paper is to answer some basic questions left unasked by Segal and Zaanen and, at the same time, attempt to throw some light upon the Radon-Nikodym theorems for regular Borel measures, and upon the way some of these can be made to follow simply from their abstract counterparts.

Notation. Measures $\mu$ and $\lambda$ will be nonnegative and defined on a sigma algebra, $\Sigma$, of subsets of a set $S$. By a measurable set we shall simply mean a member of $\Sigma$. A measurable set $A$ is called $\mu$-null if $\mu(A) = 0$. If every measurable subset of $A$ with finite $\mu$ measure is $\mu$-null, then we say that $A$ is $\mu$-locally null. The nonnull locally null sets of a measure $\mu$ can be "killed" by replacing $\mu$ by its corresponding contracted measure $\mu^*$ defined by $\mu^*(A) = \sup \{\mu(B) : B \subseteq A \text{ and } \mu(B) < \infty\}$. (See Zaanen [7, §10].) Clearly, $\mu = \mu^*$ iff $\mu$ has no nonnull locally null sets. If every $\mu$-null set is $\lambda$-null, then we say that $\lambda$ is absolutely continuous with respect to $\mu$ and write $\lambda \ll \mu$. The relation $d\lambda = g \, d\mu$ is said to hold
\(\mu\)-locally (respectively, \(\lambda\)-locally) in a set \(A\) if we have

\begin{equation}
\lambda(B) = \int_B g \, d\mu
\end{equation}

for every \(\mu\)-sigma finite (\(\lambda\)-sigma finite) subset \(B\) of \(A\). If (1) holds for every measurable subset \(B\) of \(A\), then the relation \(d\lambda = g \, d\mu\) is said to hold \textit{globally} in \(A\). If \(A\) is not mentioned then it will be understood that \(A = \mathcal{S}\). The minimum of measures \(\mu\) and \(\lambda\) (in the lattice of nonnegative measures defined on \(\Sigma\)) will be denoted as usual by \(\mu \wedge \lambda\). A \textit{cross section} of a measure \(\mu\) is a family \(\{g_A : \mu(A)\text{ is sigma finite}\}\) where for each \(A\), \(g_A\) is a nonnegative measurable function which vanishes \(\mu\)-almost everywhere outside \(A\), and such that whenever \(A\) and \(B\) are \(\mu\)-sigma finite sets, the functions \(g_A\) and \(g_B\) agree \(\mu\)-almost everywhere in \(A \cap B\).

**Theorem 1.1** (The classical Radon-Nikodym theorem). If \(\lambda \ll \mu\) and \(\mu\) is sigma finite, then there exists a measurable function \(g\) such that \(d\lambda = g \, d\mu\) globally.

Note that \(\lambda\) need not be sigma finite (Halmos [1, p. 131]). From this theorem the following general result follows easily (Zaanen [7, Theorem 7.3]).

**Theorem 1.2.** If \(\lambda \ll \mu\), then there exists a cross section \(\{g_A\}\) of \(\mu\) such that whenever a set \(A\) is \(\mu\)-sigma finite, we have \(d\lambda = g_A \, d\mu\) globally in \(A\).

The work of Segal and Zaanen is largely concerned with the question as to when every cross section \(\{g_A\}\) of a given measure \(\mu\) can be pieced together into a single measurable function \(g\) which will agree \(\mu\)-almost everywhere with each function \(g_A\) on the corresponding set \(A\). They show that this property of \(\mu\) (called the R-N property) is equivalent to the condition that given any \(\lambda\) satisfying \(\lambda \ll \mu\), there exists a measurable function \(g\) such that \(d\lambda = g \, d\mu\) \(\mu\)-locally. Among the other equivalent conditions that they give are Segal's well-known criterion of localizability and the condition that \(L^1(\mu)^* = L^\infty(\mu)\). In addition, they show that if \(\mu\) is decomposable into finite measures (Hewitt and Stromberg [3, \S\S 19.25 and 19.27], and Zaanen [7, Theorem 7.1]), then \(\mu\) has the R-N property. Of course, every sigma finite measure has the R-N property.

However, Segal and Zaanen do not give conditions under which the relation \(d\lambda = g \, d\mu\) will be valid \textit{globally}, and it is with this question that we are concerned here.
2. Compatibility. If \( \lambda \) has no nonnull locally null sets, then whenever 
\( \lambda(A) = \infty \), \( A \) must contain subsets of arbitrarily large finite \( \lambda \) measure, and consequently, if for a given \( \mu \), the relation 
\( d\lambda = g \, d\mu \) holds \( \lambda \)-locally, it 
actually holds globally. However, even if neither \( \mu \) nor \( \lambda \) has nonnull 
locally null sets, the relation 
\( d\lambda = g \, d\mu \) can quite easily hold \( \mu \)-locally 
without holding globally. We may look, for instance, at the famous "Saks 
example" in which \( \lambda \) is Lebesgue measure and \( \mu \) is counting measure on 
the measurable subsets of \([0, 1]\). In this example we see that 
\( d\lambda = 0 \, d\mu \) 
\( \mu \)-locally, but that there is no function \( g \) such that 
\( d\lambda = g \, d\mu \) globally.

Definition. We say that \( \lambda \) is compatible with \( \mu \) if whenever 
\( 0 < \lambda(A) < \infty \), there exists a measurable subset \( B \) of \( A \) such that 
\( \lambda(B) > 0 \) and 
\( \mu(B) < \infty \).

Note that any measure is compatible with a sigma finite measure. The 
following theorem is clear:

Theorem 2.1. A necessary condition for the existence of a measurable 
function \( g \) such that 
\( d\lambda = g \, d\mu \) globally is that \( \lambda \ll \mu \) and that \( \lambda \) be com-
patible with \( \mu \).

The converse of Theorem 2.1 does not hold, as we see in the following 
three examples:

Example 1. (Cf. Zaanen [7, Example 4] or Royden [4, p. 249, Problem 
46].) Let \( M \) and \( N \) be uncountable sets such that 
\( |M| < |N| \), and let \( S = M \times N \). Call a set \( A \) measurable if, for every horizontal or vertical line \( L \) 
in \( M \times N \), either \( L \cap A \) or \( L \setminus A \) is countable. For each measurable \( A \), let 
\( \mu(A) \) be the number of horizontal or vertical lines \( L \) such that 
\( L \setminus A \) is 
countable, and define \( \lambda(A) \) similarly, counting the horizontal lines only.
Then although neither \( \mu \) nor \( \lambda \) has nonnull locally null sets and \( \lambda \) is com-
patible with \( \mu \) and \( \lambda \ll \mu \), there is no function \( g \) such that 
\( d\lambda = g \, d\mu \) 
globally, nor even a \( g \) such that 
\( d\lambda = g \, d\mu \) \( \mu \)-locally. Of course, \( \mu \) does not 
have the \( R-N \) property.

Example 2. Let \( R_d \) be the discrete group of reals and let \( \lambda \) be Haar 
measure on the locally compact group \( R_d \times R \). Define \( \mu \) to be \( \lambda \) plus the 
counting measure on \( R_d \times |0| \). Then \( \lambda \) is compatible with \( \mu \) and \( \lambda \ll \mu \), 
and although it is easy to find \( g \) such that 
\( d\lambda = g \, d\mu \) \( \mu \)-locally, there is no 
g such that 
\( d\lambda = g \, d\mu \) globally.

Example 3. Let \( \mu \) be Haar measure on \( R_d \times R \) and let \( \lambda = \mu^* \). Then 
\( \lambda \) is compatible with \( \mu \), \( \lambda \ll \mu \), \( \lambda \) has no nonnull locally null sets, and 
\( d\lambda = 1 \, d\mu \) \( \mu \)-locally, but there is no \( g \) such that 
\( d\lambda = g \, d\mu \) globally.
If \( \lambda \) is finite, the pathology of the above examples disappears.

**Theorem 2.2.** If \( \lambda(S) < \infty, \lambda \ll \mu, \) and \( \lambda \) is compatible with \( \mu \), then there exists a measurable function \( g \) such that \( d\lambda = g \, d\mu \) globally.

**Proof.** Define \( \alpha = \sup \{ \lambda(A) : \mu(A) \text{ is sigma finite} \} \), and choose an increasing sequence \( \{ A_n \} \) of measurable sets such that \( \lambda(A_n) \to \alpha \) as \( n \to \infty \), and define \( A = \bigcup_{n=1}^{\infty} A_n \). Then \( \lambda(A) = \alpha \) and \( A \) is \( \mu \)-sigma finite. Clearly \( \lambda(S \setminus A) = 0 \), for otherwise there would exist a measurable subset \( B \) of \( S \setminus A \) such that \( \lambda(B) \neq 0 \) and \( \mu(B) < \infty \), and we would have \( \lambda(A \cup B) > \alpha \), in spite of the fact that \( \mu(A \cup B) \) is sigma finite. Using the classical Radon-Nikodym theorem, choose a measurable function \( g \) such that \( d\lambda = g \, d\mu \) holds globally inside \( A \), and define \( g(x) = 0 \) for every \( x \) outside \( A \). Clearly, \( d\lambda = g \, d\mu \) globally. This completes the proof.

In the Saks example (above), \( \lambda \) fails to be compatible with \( \mu \) in a rather strong way. In fact, \( \lambda \) is totally incompatible with \( \mu \), by which we mean that \( \lambda(A) \neq 0 \Rightarrow \mu(A) = 0 \). By returning to one of the classical proofs of the Radon-Nikodym theorem, we can improve Theorem 2.2.

**Theorem 2.3.** Suppose \( \lambda \) is finite. Then \( \lambda \) can be uniquely decomposed into the sum of three measures \( \lambda_s, \lambda_t \), \( \lambda_c \), where \( \lambda_s \) is singular to \( \mu \), \( \lambda_t \) is totally incompatible with \( \mu \), and \( \lambda_c \) is both absolutely continuous and compatible with \( \mu \). A fortiori there exists \( g \in L^1(\mu) \) such that \( \lambda(A) = \lambda_s(A) + \lambda_t(A) + \int_A g \, d\mu \) for every measurable set \( A \).

**Proof.** Define \( \phi = \lambda + \mu \), and let \( T \) be the functional on \( L^2(\phi) \) defined by \( T(f) = \int_S f \, d\lambda \) for every \( f \in L^2(\phi) \). Then since

\[
|T(f)|^2 \leq \left| \int_S f \, d\lambda \right|^2 \leq \lambda(S) \int_S |f|^2 \, d\lambda \leq \lambda(S) \int_S |f|^2 \, d\phi,
\]

we see that \( \|T\| \leq \sqrt{\lambda(S)} \). Therefore, there exists \( h \in L^2(\phi) \) such that \( 0 \leq h \leq 1 \) and \( \int_S |f| \, d\lambda = \int_S |h| \, d\phi \) for every \( f \in L^2(\phi) \). The proof is now concluded by defining \( \lambda_s, \lambda_t \), and \( \lambda_c \) to be the respective restrictions of \( \lambda \) to the sets of points \( x \) where \( h(x) \) is, respectively, one, zero, or satisfies \( 0 < h(x) < 1 \), and defining \( g = (h/(1-h))\chi_{C} \), where \( C = \{x : 0 < h(x) < 1\} \).

It follows from these theorems that even if the measure \( \mu \) fails to have the R-N property, we can nevertheless obtain the relation \( d\lambda = g \, d\mu \) globally when \( \lambda \) is finite, \( \lambda \ll \mu \) and \( \lambda \) is compatible with \( \mu \). Furthermore, we obtain

**Corollary 2.4.** Let \( T \) be a bounded linear functional on \( L^1(\mu) \), and suppose there exists a constant \( K \) such that \( |T(f)| \leq K\|f\|_{\infty} \) for every
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Then there exists $g \in L^\infty(\mu)$ such that $T(f) = \int g \, d\mu$ for every $f \in L^1(\mu)$.

**Proof.** Note that by $||f||_\infty$ we mean the least number $\alpha$ such that $\{x : f(x) > \alpha\}$ is $\mu$-locally null. We may assume $T \geq 0$. Now for every measurable set $A$ define $\lambda(A) = \sup \{T(\chi_B) : B \subseteq A$ and $\mu(B) < \infty\}$. Then the measure $\lambda$ is finite, $\lambda \ll \mu$ and $\lambda$ is compatible with $\mu$, and so by Theorem 2.2 there exists $g$ such that $d\lambda = g \, d\mu$ globally, and it is easy to see that the function $g$ has all the required properties.

3. The case $\lambda$ infinite. We have remarked that if $\lambda$ is infinite, then Theorem 2.1 has no general converse. Positive results may be obtained, however, by imposing certain other restrictions on $\mu$ and $\lambda$.

**Theorem 3.1.** Suppose neither $\lambda$ nor $\mu$ has nonnull locally null sets, $\lambda \ll \mu$, $\lambda$ is compatible with $\mu$, and that there exists a measurable function $g$ such that $d\lambda = g \, d\mu$ $\mu$-locally. (The latter condition would hold, for example, if $\mu$ had the R-N property.) Then for any such function $g$ we have $d\lambda = g \, d\mu$ globally.

**Proof.** We need only show that $d\lambda = g \, d\mu$ locally. Suppose $A(A) < \infty$ and, using Theorem 2.2, choose a function $h$ defined in $A$ such that $d\lambda = h \, d\mu$ globally in $A$. Clearly $h$ and $g$ agree $\mu$-locally almost everywhere, and therefore $\mu$-almost everywhere, in $A$. Hence $\lambda(A) = \int_A h \, d\mu = \int_A g \, d\mu$, and the result follows.

**Theorem 3.2.** Suppose neither $\lambda$ nor $\mu$ has nonnull locally null sets, $\lambda \ll \mu$, $\lambda$ is compatible with $\mu$, and that $\lambda$ has the R-N property. Then there exists a measurable function $g$ such that given any measurable set $A$, there exists a measurable subset $B$ of $A$ such that $\lambda(A) = \lambda(B) = \int_B g \, d\mu$.

**Proof.** For every set $A$ of sigma finite $\lambda$ measure, choose a measurable function $g_A$ which vanishes outside $A$ and satisfies $d\lambda = g_A \, d\mu$ globally in $A$ (see Theorem 2.2). Note that if $A$ and $B$ are two sets of sigma finite $\lambda$ measure, then $g_A$ and $g_B$ clearly agree $\mu$-locally almost everywhere, and therefore $\mu$-almost everywhere, in $A \cap B$. Therefore since $\lambda \ll \mu$ we see that $g_A$ and $g_B$ agree $\lambda$-almost everywhere in $A \cap B$, and it follows that $\{g_A : \lambda(A) \text{ is sigma finite}\}$ is a cross section of $\lambda$, and so there exists a measurable function $g$ such that whenever $\lambda(A)$ is sigma finite, $g$ and $g_A$ agree $\lambda$-almost everywhere in $A$. 

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Now suppose that \( \lambda(A) < \infty \), and define \( B = \{ x \in A : g_A(x) = g(x) \} \). Then \( \lambda(A) = \lambda(B) = \int_B g_A \, d\mu = \int_B g \, d\mu \), and, in particular, \( \lambda(A) \leq \int_A g \, d\mu \).

On the other hand, if \( \lambda(A) = \infty \), we see that \( A \) contains subsets \( B \) of arbitrarily large finite \( \lambda \) measure, and we obtain
\[
\int_A g \, d\mu \geq \sup \left\{ \int_B g \, d\mu : B \subseteq A \text{ and } \lambda(B) < \infty \right\}
\geq \sup \{ \lambda(B) : B \subseteq A \text{ and } \lambda(B) < \infty \} = \infty.
\]
This completes the proof.

Roughly speaking, small measures are more likely to have the R-N property than large ones. Therefore the following combination of Theorems 3.1 and 3.2 is of interest.

**Theorem 3.3.** Suppose neither \( \lambda \) nor \( \mu \) has nonnull locally null sets, \( \lambda \ll \mu \), \( \lambda \) is compatible with \( \mu \), and \( \mu \wedge \lambda \) has the R-N property. Then there exists a measurable function \( g \) such that the conclusions of Theorem 3.2 hold.

**Proof.** The result follows on combining Theorems 3.1 and 3.2, after noticing that \( \lambda \ll \mu \wedge \lambda \), and \( \mu \wedge \lambda \) clearly has no nonnull locally null sets.

4. Regular Borel measures. By a regular Borel measure on a locally compact space \( S \) we mean a measure \( \mu \) constructed from a nonnegative functional as in Rudin [5, Theorem 2.14], or Hewitt and Ross [2, Chapter 11], or Hewitt and Stromberg [3, \S\S 9 and 10]. \( \mu \) is really an outer measure defined on all subsets of \( S \), and is a measure on the family \( \Sigma_\mu \) of \( \mu \)-measurable sets which contains all Borel sets. Note that if \( \lambda \) and \( \mu \) are regular Borel measures, and every \( \mu \)-null set is \( \lambda \)-locally null (for example, we might have \( \lambda \ll \mu \)), then \( \Sigma_\mu \subset \Sigma_\lambda \), so we may regard both \( \lambda \) and \( \mu \) as defined on the same sigma algebra \( \Sigma_\mu \). Furthermore, \( \lambda \) is automatically compatible with \( \mu \), for if \( 0 < \lambda(A) < \infty \), then there is a compact subset \( K \) of \( A \) such that \( \lambda(K) > 0 \), and of course \( \mu(K) < \infty \). Finally, we note from Hewitt and Stromberg [3, Corollary 19.31] that a regular Borel measure has the R-N property.\(^1\)

From these observations, we obtain immediate proofs of the following three theorems (the first two of which are well known).

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\(^1\)In saying that a regular Borel measure has the R-N property, we regard \( \mu \) as defined on all \( \mu \)-measurable sets, not only on the Borel sets.
Theorem 4.1 (Hewitt and Stromberg [3, Theorem 19.32]). Let \( \mu \) be a regular Borel measure, let \( \lambda \) be any measure defined on \( \Sigma_\mu \) and suppose \( \lambda \ll \mu \). Then there is a \( \mu \)-measurable function \( g \) such that \( d\lambda = g\,d\mu \) \( \mu \)-locally.

Proof. The result follows at once since \( \mu \) has the R-N property.

Theorem 4.1 is essentially the same result as Hewitt and Ross [2, Theorem 12.17]. (It really is!)

Theorem 4.2 (Hewitt and Ross [2, Theorem 14.19]). Let \( \lambda \) and \( \mu \) be regular Borel measures, suppose \( \lambda \ll \mu \) and suppose \( \lambda(S) < \infty \). Then there is a \( \mu \)-measurable function \( g \) such that \( d\lambda = g\,d\mu \) globally.

Proof. The result follows at once from Theorem 2.2 since \( \lambda \) is compatible with \( \mu \).

The following result follows at once from Theorem 3.1.

Theorem 4.3. If \( \lambda \) and \( \mu \) are regular Borel measures and neither \( \lambda \) nor \( \mu \) has nonnull locally null sets and \( \lambda \ll \mu \), there is a \( \mu \)-measurable function \( g \) such that \( d\lambda = g\,d\mu \) globally.

Actually, one can do better than this, but the sharper result must be proved directly:

Theorem 4.4. Let \( \lambda \) and \( \mu \) be regular Borel measures and suppose that every \( \mu \)-null set is \( \lambda \)-locally null. Then there is a \( \mu \)-measurable function \( g \) such that \( d\lambda = g\,d\mu \) \( \lambda \)-locally.

Proof. Choose a family \( \mathcal{F} \) of compact sets for the measure \( \lambda \) as in Hewitt and Ross [2, Theorem 11.39]. Let \( N = S \setminus \bigcup \mathcal{F} \). Then \( N \) is \( \lambda \)-locally null. For each \( F \in \mathcal{F} \) the measure \( \lambda_F \), defined by \( \lambda_F(A) = \lambda(A \cap F) \) for all \( \Sigma_\mu \)-measurable sets \( A \), is a finite measure compatible with \( \mu \), and so by Theorem 2.2 there is a \( \mu \)-measurable function \( g_F \) such that \( d\lambda_F = g_F\,d\mu \) globally. We may assume that \( g_F \) is zero outside \( F \). Define \( g \) as follows: If \( x \in N \) then \( g(x) = 0 \), and if \( x \in F \in \mathcal{F} \) then \( g(x) = g_F(x) \). Then \( g \) is \( \mu \)-measurable: (This may be a little surprising, since there is no reason to expect \( N \in \Sigma_\mu \).) For if \( \alpha < 0 \) then \( \{x: g(x) > \alpha\} = S \), and if \( \alpha \geq 0 \), then \( \{x: g(x) > \alpha\} \) is \( \mu \)-measurable since its intersection with any compact set \( K \) is of the form \( \bigcup_{n=1}^\infty K \cap \{x: g_F(x) > \alpha\} \in \Sigma_\mu \). Clearly \( d\lambda = g\,d\mu \) \( \lambda \)-locally.

This gives the ultimate result for regular Borel measures:
Corollary 4.5. Let $\lambda$ and $\mu$ be regular Borel measures, suppose $\lambda$ has no nonnull locally null sets and $\lambda \ll \mu$. Then there exists a $\mu$-measurable function $g$ such that $d\lambda = g\,d\mu$ globally.

Remark. Example 2 shows that in Corollary 4.5 we cannot dispense with the assumption that $\lambda$ has no nonnull locally null sets.

Example 3 shows that in Theorem 4.4 and Corollary 4.5 we cannot dispense with the assumption that $\lambda$ be regular. It also shows that in Theorem 3.1 we cannot dispense with the assumption that $\mu$ have no nonnull locally null sets.

Note however, that in Theorem 4.4 and Corollary 4.5, $\mu$ may have non-null locally null sets.

REFERENCES


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