

## AN EXAMPLE OF NONEXCISIVENESS IN SHEAF COHOMOLOGY

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**ABSTRACT.** An example is constructed to show that the conditions of tautness, besides others, imposed on spaces in a result on  $\phi$ -excisive couples obtained by Bredon [1] are needed.

**Introduction.** Let  $X_1, X_2$  be a pair of subspaces of a topological space  $X$ ; and  $\phi$  a family of supports on  $X$ . Bredon [1, Definition 13.1, p. 65] calls the pair  $(X_1, X_2)$   $\phi$ -excisive with respect to sheaf-theoretic relative cohomology groups, defined by him [1, p. 57], if the inclusion map  $i: (X_1, X_1 \cap X_2) \rightarrow (X_1 \cup X_2, X_2)$  induces isomorphism

$$i^*: H_{\phi \cap (X_1 \cup X_2)}^*(X_1 \cup X_2, X_2; \mathcal{Q}) \rightarrow H_{\phi \cap X_1}^*(X_1, X_1 \cap X_2; \mathcal{Q})$$

for any sheaf  $\mathcal{Q}$  on  $X$ . Among other things he has proved the following

**Proposition** [1, 13.3, p. 68]. *Let  $X = X_1 \cup X_2$  and  $A = X_1 \cap X_2$ . If  $\phi$  is a family of supports on  $X$  with*

$$(*) \quad (\phi \cap X_1) |_{X_1 - A} = \phi |_{(X - X_2)} \subset \phi |_{\text{Int}(X_1)}$$

*and such that  $X_2$  is  $\phi$ -taut and  $A$  is  $(\phi \cap X_1)$ -taut in  $X_1$ , then  $(X_1, X_2)$  is  $\phi$ -excisive.*

and a consequent,

**Corollary.** *If  $X = X_1 \cup \text{Int}(X_2) = \text{Int}(X_1) \cup X_2$  and  $X_1, X_2, X_1 \cap X_2$  are all  $\phi$ -taut in  $X$ , then  $(X_1, X_2)$  is  $\phi$ -excisive.*

At the end of the proof he raises a problem: Are the tautness conditions in the above Proposition and Corollary needed? We give an example to show that the answer to the case of the Proposition is affirmative. The case of the Corollary is still open. His result is of interest not only for sheaf cohomology, but it also enlarges the class of excisive couples [2, p. 188] for Alexander-Spanier cohomology, as we shall see later, and consequently,

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the validity of Mayer-Vietoris exact sequences. Our example uses a sheaf of abelian groups as the simple sheaf  $\mathbf{R}$  of real numbers on a metric space  $X$ . For all definitions, notations and results indirectly involved, [1] is our complete reference.

**Example.** Let  $X = \mathbf{R}^2$  be the 2-Euclidean space and  $\mathbf{R}$  be the constant sheaf of real numbers on  $X$ . Define

$$\begin{aligned} X_1 &= \{(x, y) \mid x^2 + y^2 \leq 1 \text{ or } y \geq 0\}, \\ X_2 &= \{(x, y) \mid x^2 + y^2 > 1 \text{ and } y \leq 0\} \cup \{(1, 0), (-1, 0)\}, \\ D &= \{(x, y) \mid x^2 + y^2 < 1\}, \\ B &= \{(x, y) \mid x^2 + y^2 = 1 \text{ and } y < 0\}, \\ \phi &= \text{cl } d \mid X - (D \cup B). \end{aligned}$$

First of all let us observe that the family  $\phi$  of supports is not paracompactifying because the closed line segment  $[(0, 1), (0, 2)] \in \phi$  but has no neighborhood which is a member of  $\phi$ . Since  $X_1$  is closed, it is clear that (\*) holds. It may be noted that  $\phi \cap X_1 = \phi \mid X_1$  and  $\phi \cap A = \phi \mid A$ . Now since  $H^0_\phi(X)$  is simply the group of those real valued functions which are locally constant and have their support in  $\phi$ , and because  $X$  is connected,  $H^0_\phi(X) = 0$ . By a similar argument,  $H^0_{\phi \cap X_2}(X_2) = 0 = H^0_{\phi \cap X_1}(X_1)$ . But, since  $A \in \phi$  and has two components,  $H^0_{\phi \cap A}(A) = \mathbf{R} \oplus \mathbf{R}$ .

Next we claim that  $X$  is  $\phi$ -contractible to any point of  $D \cup B$ . To see why it is  $\phi$ -contractible to the origin  $(0, 0)$ , consider the  $\phi$ -proper maps  $I_X$  and the constant map  $C: X \rightarrow X$  with value  $(0, 0)$ . Define the usual homotopy  $F: X \times I \rightarrow X$  by  $F((x, y), t) = ((1 - t)x, (1 - t)y)$ . Then for any  $K \in \phi$ ,

$$F^{-1}(K) = \{(x, y), t \mid ((1 - t)x, (1 - t)y) \in K\} = \bigcup_{0 \leq t < 1} S_t \times t,$$

where

$$S_t = \{(x, y) \mid (x, y) \in K/(1 - t)\}.$$

This is a closed subset of  $S \times I \in \phi \times I$ , where  $S$  is the union of all rays starting from each point of  $K$  away from the origin. Hence by the homotopy axiom for constant sheaves [1, Theorem 11.4, p. 56],  $H^n_\phi(X) = 0$  for every  $n \geq 0$ .

Finally, one observes that  $(X_1, X_2)$  cannot be  $\phi$ -excisive because the Mayer-Vietoris sequence [1, p. 68]

$$0 \rightarrow H^0_\phi(X) \rightarrow H^0_{\phi \cap X_1}(X_1) \oplus H^0_{\phi \cap X_2}(X_2) \rightarrow H^0_{\phi \cap A}(A) \rightarrow H^1_\phi(X) \rightarrow \dots$$

is no longer exact.

Now to see that  $A$  is not  $\phi \cap X_1$ -taut in  $X_1$ , consider the locally constant function  $f: A \rightarrow R$  with any nonzero value, which is in fact an element of  $H_{\phi \cap A}^0(A)$ . Then this has no extension  $f'$  to a neighborhood  $N$  of  $A$  in  $X_1$  which has its support in  $\phi \cap N$ , i.e.,

$$\eta: \varinjlim H_{\phi \cap N}^0(N) \rightarrow H_{\phi \cap A}^0(A)$$

is not epic, and our assertion follows from [1, Theorem 10.5, pp. 52–53].

We are unable to see whether or not each of the tautness conditions is separately needed.

**Remark.** The conclusion that  $(X_1, X_2)$  is not  $\phi$ -excisive depends solely on the fact that  $\phi$  is not a paracompactifying family of supports, since otherwise  $X_1, X_2$  being arbitrary subsets of a metric space would be  $\phi$ -taut. For example, when  $\phi = \text{cl } d$ , all the groups are Alexander-Spanier cohomology groups, and in that case  $(X_1, X_2)$  is  $\phi$ -excisive by the Proposition itself. This very last statement illustrates the situation about excisive couples not covered under the excisive couples of Alexander-Spanier or Čech cohomology theories.

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#### REFERENCES

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