UNIQUE HAHN-BANACH EXTENSIONS
AND KOROVKIN'S THEOREM

LYNN C. KURTZ

ABSTRACT. This paper characterizes in terms of weak topologies those bounded linear functionals on a subspace which have unique Hahn-Banach extensions to the whole linear normed space. The relationship to the Choquet boundary is discussed, and a Korovkin type theorem is obtained.

We begin with a different proof of Korovkin's theorem in order to motivate that which follows.

Theorem 1 (Korovkin [2]). Suppose \{L_n\} is a sequence of positive linear functionals on \( C[0, 1] \) satisfying \( L_n(1) \rightarrow 1 \), \( L_n(l) \rightarrow c \), and \( L_n(l^2) \rightarrow c^2 \). (Here \( l \) is the identity function.) Then \( L_n(f) \rightarrow f(c) \) for all \( f \in C[0, 1] \).

Proof of Theorem 1. Identify \( C^* \) with the regular Borel measures on \([0, 1]\). Identify \( L_n \) with a regular positive measure by \( L_n(f) = \int f d\mu_n \). Any subnet \( \mu_\alpha \) of \( \{\mu_n\} \) has a further subnet \( \mu_\gamma \) converging \( w(C^*, C) \) to some \( \mu \), and it follows that \( \mu_\gamma(1) \rightarrow \mu(1) = \|\mu\| \). In particular \( \mu(1) = 1 \), \( \mu(l) = c \), and \( \mu(l^2) = c^2 \). Then \( \mu \) must be \( \delta_c \), the point evaluation measure, for suppose \( \mu(1) < 1 \); then there is an open interval \( J \) containing \( c \) with \( \mu(J) < 1 \). Let \( g(x) = (x - c)^2 \) and \( \tau = \min J g(x) > 0 \). Then

\[
0 = \int_0^1 (l - c)^2 d\mu \geq \int_0^1 (l - c)^2 d\mu = \tau \mu(\hat{f}) = \tau (1 - \mu(f)) > 0,
\]

a contradiction. Since every subnet of \( \{\mu_n\} \) has a further subnet converging \( w(C^*, C) \) to \( \delta_c \), then \( \mu_n \rightarrow \delta_c \) in \( w(C^*, C) \), hence \( L_n f = \int f d\mu_n \rightarrow \int d\delta_c = f(c) \) for all \( f \in C[0, 1] \).

While this proof is admittedly less elementary than Korovkin's, it makes it clear that the heart of the matter is the fact that the point evaluation func-
tional $\xi_c$ is the only Hahn-Banach extension of its restriction to the subspace $\text{span}(1, I, I^2)$. This, of course, can be viewed as resulting from the fact that the Choquet boundary of $\text{span}(1, I, I^2)$ is $[0, 1]$. Wulbert [7], [8], using this observation and the fact that point evaluations are the extreme points in the unit ball of $C^*$, has defined a generalized Choquet boundary $cb(P)$ for weakly separating subspaces $P$ of a linear normed space, and he proved a nice Korovkin theorem for such subspaces. In this paper we take the position that the central issue is not extreme points, but unique Hahn-Banach extensions.

If $(X, T_1, T_2)$ is a space with two topologies, we shall say the topologies $T_1$ and $T_2$ are equivalent at the point $x$ if they have equivalent neighborhood bases at $x$. This can happen if and only if each net $x_\delta$ in $X$ converging to $x$ in $T_1$ converges to $x$ in $T_2$ and vice versa. Also recall that if $S \subset X$, $x_\delta$ is a net in $S$, and $x \in S$, then $x_\delta \rightarrow x$ in the relative $T_1$ topology on $S$ if and only if $x_\delta \rightarrow x$ in $(X, T_1)$ [6].

In what follows, if $X$ is a linear normed space, we denote by $X^*$ the space of continuous linear functionals on $X$, by $S^*$ the unit ball in $X^*$, and by $w(X^*, X)$ the weak-star topology on $X^*$.

**Definition.** If $X$ is a linear normed space and $M$ is a subspace, we define the set $K(M)$ by $K(M) = \{x^* \in X^* \mid \|x^*\| = 1 \text{ and such that } x^*|M = y^*|M \text{ implies } x^* = y^* \text{ if } \|y^*\| \leq 1\}$. Note that if $X = C[0, 1]$, $M$ is a subspace containing the constants and separating points of $[0, 1]$, and the points of the Choquet boundary $B(M)$ are identified with their point evaluation functionals, then $B(M) \subset K(M)$. Examples with other subspaces $M$ follow.

(i) $M = \text{span}(1)$. The value $\mu(1)$ is the $\mu$ measure of $[0, 1]$, and it is clear that any $\mu$ with $\|\mu\| = 1$ has many different Hahn-Banach extensions, so $K(M) = \Phi$.

(ii) $M = \{f/\ell_2 = (f(0) + f(1))/2\}$ (Phelps [3]). Here $B(M) = \{\xi_\alpha \mid 0 \leq \alpha \leq 1, \alpha \neq 1/2\}$. Here $B(M) \subset K(M)$ properly since any measure $\mu$ with $\|\mu\| = 1$ having the property that there is an open interval $J \subset [0, 1]$ with $1/2 \in J$ and $|\mu|(J) = 0$ is in $K(M)$. Note that $\xi_{1/2} \notin K(M)$.

(iii) If $M$ is dense, then $K(M) = \{x^* \in S^* \mid \|x^*\| = 1\}$.

We now formulate a theorem characterizing the points of $K(M)$.

**Theorem 2.** A point $x^* \in S^*$ with $\|x^*\| = 1$ is in $K(M)$ if and only if the topologies induced on $S^*$ by $w(X^*, M)$ and $w(X^*, X)$ are equivalent at $x^*$.

**Proof of Theorem 2.** Suppose $x^* \in K(M)$. Suppose $x^*_\alpha$ is a net in $S^*$ converging to $x^*$ in relative $w(X^*, M)$. Any subnet $x^*_\beta$ has a $w(X^*, X)$ (hence...
relative $w(X^*, X)$ convergent subnet $x^*_\gamma$, say $x^*_{\gamma} \rightarrow y^*$. It follows that $y^* \in S^*$ and $y^*|M = x^*|M$. Hence $y^* = x^*$ since $x^* \in K(M)$ and $x^*_\alpha \rightarrow x^*$ in relative $w(X^*, X)$. Conversely, suppose the two topologies are equivalent at $x^* \in S^*$, where $\|x^*\| = 1$. Suppose $y^* \in S^*$ and $y^*|M = x^*|M$. Define a net $x^*_{\alpha} \rightarrow y^*$ relative $w(X^*, M)$, hence in relative $w(X^*, X)$, hence in $w(X^*, X)$. Similarly $x^*_\alpha \rightarrow x^*$ relative $w(X^*, M)$, hence in $w(X^*, X)$. This implies that $x^* = y^*$, so $x^* \in K(M)$ and the proof is complete.

In [7], Wulbert proves a theorem similar to the "only if" part of the last theorem for weakly separating subspaces $M$ and the corresponding $cb(M)$. In this case $cb(M) \subseteq K(M)$; for suppose $x^* \in cb(M)$; then $x^*$ is extreme in $S(M^*)$. Suppose $y^* \in S^*$ with $y^*|M = x^*|M$. If $y^*$ is extreme in $S^*$, then $y^* = x^*$ since $M$ is weakly separating. In fact, $y^*$ must be extreme in $S^*$, for if not, then $y^*$ and $x^*$ are in the face $F = \{z^*| \|z^*\| = 1 \}$ and $z^*|M = x^*|M$, and the Krein-Milman theorem implies the existence of an extreme point $w^* \neq x^*$ in $F$. This contradicts the fact that $M$ is weakly separating.

This shows $x^* \in K(M)$ and $cb(M) \subseteq K(M)$. We also observe that if $x^* \in K(M)$ and $x^*|M$ is extreme in $S(M^*)$, then $x^*$ is extreme in $S^*$; for suppose $x^* = ty^* + (1 - t)z^*$, $y^*, z^* \in S^*$, $0 < t < 1$. Restricting this to $M$ gives $x^*|M = y^*|M = z^*|M$ since $x^*|M$ is extreme in $S(M^*)$. But then $x^* = y^* = z^*$, since $x^* \in K(M)$, hence $x^*$ is extreme in $S^*$.

We now formulate a Korovkin type theorem.

**Theorem 3.** If $L_\alpha$ is a net of operators in $B[X, X]$ with $\|L_\alpha\| \leq 1$, and if $\|L_\alpha x - x\| \rightarrow 0$ for all $x$ in a subspace $M$, then $x^* L_\alpha x \rightarrow x^* x$ uniformly on all $w(X^*, X)$ compact subsets of $K(M)$ for each $x$ in $X$.

**Proof of Theorem 3.** If not, there exist $\epsilon > 0$, a $w(X^*, X)$ compact set $D \subseteq K(M)$, $x \in X$, a subnet $L_\beta$ of $L_\alpha$, and a net $x^*_\beta \in D$ such that

\[
|x^*_\beta L_\beta x - x^* x| \geq \epsilon.
\]

By taking a further subnet, we may assume $x^*_\beta \rightarrow x^* \in K(M)$ in $w(X^*, X)$. If $y \in M$, then

\[
|x^*_\beta L_\beta y - x^* y| \leq |x^*_\beta L_\beta y - x^*_\beta y| + |x^*_\beta y - x^* y|
\leq \|x^*_\beta\| \|L_\beta y - y\| + \|x^*_\beta y - x^* y\| \rightarrow 0.
\]

But since $x^* \in K(M)$ and $\|x^*_\beta L_\beta\| \leq \|x^*_\beta\| \|L_\beta\| \leq 1$, this implies $x^*_\beta L_\beta(x) \rightarrow x^*(x)$ for all $x \in X$. This together with the fact that $x^*_\beta(x) \rightarrow x^*(x)$ for all $x \in X$ contradicts inequality (*) above, and completes the proof.

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While this argument is much the same as Wulbert's [8], it should be
noted we have no assumptions on \( M \) other than it is a subspace, and \( cb(M) \subset K(M) \) can be a proper containment when \( cb(M) \) exists.

REFERENCES


DEPARTMENT OF MATHEMATICS, ARIZONA STATE UNIVERSITY, TEMPE, ARIZONA 85281 (Current address)

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF KENTUCKY, LEXINGTON, KENTUCKY 40506