THE MEASURE OF THE INTERSECTION OF ROTATES
OF A SET ON THE CIRCLE \(^1\)

WOLFGANG M. SCHMIDT

ABSTRACT. Let \(S\) be a set of real numbers modulo 1 of Lebesgue
measure less than 1. It is shown that for every \(\epsilon > 0\) and for large \(k\),
there exist translates \(S + y_1, \ldots, S + y_k\) of \(S\) such that the measure of
their intersection is less than \(\epsilon k\).

1. Let \(U\) be the group of real numbers modulo 1, and \(S\) a subset of
Lebesgue measure \(\mu(S) < 1\). Given real numbers \(y_1, \ldots, y_k\), write
\(\mu(y_1, \ldots, y_k)\) for the measure of the intersection of the \(k\) translates \(S + y_1, S + y_2, \ldots, S + y_k\). Finally, denote by \(\phi(k)\) the infimum of
\(\mu(y_1, \ldots, y_k)\) over all \(k\)-tuples \(y_1, \ldots, y_k\). Erdős, Rubel and Spencer\(^2\)
had conjectured that
\[
\lim_{k \to \infty} \phi(k)^{1/k} = 0.
\]
In the present note we shall prove this conjecture.

The convergence expressed by (1) is not uniform with respect to the sets
\(S\). In fact, it can be shown that for \(0 < \alpha < 1, \epsilon > 0\) and \(k \geq 1\), there exist
sets \(S\) with \(\mu(S) = \alpha\) and \(\phi(k)^{1/k} > \alpha - \epsilon\).

2. Since \(\mu(S) < 1\), the set \(S\) is contained in a countable union of intervals
whose total measure is less than 1. In fact, this is true even with intervals
of the type \(a < x < b\)\(^3\) with rational endpoints \(a, b\). Hence we may assume
that \(S\) itself is a countable union of such intervals.

Using the easily established relation
\[
\int_U \mu(y_1, \ldots, y_m, z_1 + x, \ldots, z_n + x) \, dx = \mu(y_1, \ldots, y_m) \mu(z_1, \ldots, z_n),
\]
Received by the editors October 6, 1972 and, in revised form, January 11, 1974.
AMS (MOS) subject classifications (1970). Primary 10F40; Secondary 28-00,
43A05.

Key words and phrases. Numbers modulo 1, measure, translates.

\(^1\) Research supported by NSF GP-33026X.
number theory conference in Colorado.

\(^3\) We are considering intervals modulo 1. Hence if \(\{x\}\) denotes the fractional
part of \(x\), the interval \(a < x < b\) consists of numbers \(x\) modulo 1 with \(\{x - a\} <
\{x - b\}.\)
one sees that $\phi(m + n) \leq \phi(m)\phi(n)$. Hence if $t$ is any positive integer, we have $\phi(jt) \leq \phi(t)^j$ ($j = 1, 2, \cdots$), and if $k$ is a large integer with $jt < k \leq (j + 1)t$, then $\phi(k) \leq \phi(jt)\phi(k - jt) \leq \phi(jt) \leq \phi(t)^j$ and

$$\phi(k)^{1/k} \leq \phi(t)^{j/k} \leq \phi(t)^{(1/t) - (1/k)}.$$  

Therefore the limit superior of $\phi(k)^{1/k}$ as $k \to \infty$ cannot exceed $\phi(t)^{1/t}$. Thus in order to prove (1), it will suffice to show that for every $\epsilon > 0$ there is an integer $t$ with

$$\phi(t)^{1/t} < \epsilon.$$

3. Write $\mu(S) = \mu$, and choose $\delta > 0$ so small that

$$2\delta < 1 - \mu \quad \text{and} \quad (\delta/(1 - \mu - \delta))^{1 - \mu} - \delta < \epsilon.$$  

We may write $S = S_1 \cup S_2$, where $S_1$ is a finite union of intervals $a < x < b$ with rational endpoints, and where $\mu(S_2) < \delta$.

Let $r$ be a common denominator of the endpoints of the intervals contributing to $S_1$. Choose an integer $s$ with $s > 1/\delta$, and put

$$t = rs, \quad \nu = 1/t.$$  

Let $\chi(x)$ be the characteristic function of $S$, and write

$$I_{\nu}(y) = \int_{y}^{y + \nu} \chi(x) \, dx.$$  

Lemma. The function

$$J_{\nu}(z) = I_{\nu}(z + \nu)I_{\nu}(z + 2\nu) \cdots I_{\nu}(z + t\nu)$$  

satisfies $J_{\nu}(z) \leq (ev)^t$.

To prove the Lemma, we observe that $S_1$ consists of a finite number (in fact less than $r$) intervals $E$ of the type $(u/r) \leq x < (u + 1)/r$ with integral $u$. For each such interval $E$ contained in $S_1$, let $E'$ be the enlarged interval $(u/r) - (1/t) \leq x < (u + 1)/r$. Let $S_1'$ be the union of the intervals $E'$ so obtained. It is clear that

$$\text{if } x + w \in S_1 \text{ with } 0 \leq w \leq \nu, \text{ then } x \in S_1'.$$

For each interval $E$ above we have $\mu(E') = \mu(E) + (1/t)$, and hence we have $\mu(S_1') < \mu(S_1) + (r/t) = \mu(S_1) + (1/s) < \mu(S) + \delta = \mu + \delta$. Now $S_1'$ is a disjoint union of intervals $(\nu/t) \leq x < (\nu + 1)/t$ with integral $\nu$. If, say, it is a disjoint union of $p$ such intervals, then $\mu(S_1') = p/\nu$ and hence

$$p = t\mu(S_1') < t(\mu + \delta).$$  

License or copyright restrictions may apply to redistribution; see http://www.ams.org/journal-terms-of-use
Exactly \( q = t - p \) of the numbers \( z + \nu, z + 2\nu, \ldots, z + t\nu \) lie outside \( S_1' \); let these be the numbers \( z + m_1\nu, z + m_2\nu, \ldots, z + m_q\nu \). Since each integral \( I_\nu(y) \) is always \( \leq \nu \), we have

\[
(7) \quad J_\nu(z) \leq \nu^p I_\nu(z + m_1\nu) \cdots I_\nu(z + m_q\nu).
\]

Now \( I_\nu(z + m_i\nu) \) is the integral of \( \chi(x) \) over the interval \( z + m_i\nu \leq x < z + (m_i + 1)\nu \) \((i = 1, \ldots, d)\). These intervals are disjoint from each other. Furthermore, since \( z + m_i\nu \not\in S_1' \), (5) implies that these intervals are disjoint from \( S_1 \). Therefore if \( S_1 \) is the complement of \( S_1 \), we have

\[
I_\nu(z + m_1\nu) + \cdots + I_\nu(z + m_q\nu) \leq \int_{S_1} \chi(x) \, dx \leq \mu(S_2) < \delta.
\]

By the arithmetic-geometric inequality, the product of the \( q \) integrals on the left is \( < (\delta/q)^q \), and (7) yields

\[
J_\nu(z) < \nu^p (\delta/q)^q = \nu^t (\delta t/q)^q.
\]

From (6) we have \( q = t - p > t(1 - \mu - \delta) \), whence

\[
(\delta t/q)^q < (\delta/(1 - \mu - \delta))^q < (\delta/(1 - \mu - \delta))^{t(1 - \mu - \delta)} < \epsilon^t
\]

by (3), and the Lemma is proved.

4. The desired inequality (2) follows at once from the Lemma by observing that

\[
\phi(t) \leq \nu^{-t} \int_{\nu}^{2\nu} dy_1 \cdots \int_{t\nu}^{(t+1)\nu} dy_t \mu(-y_1, \ldots, -y_t)
\]

\[
= \nu^{-t} \int_U dx \int_{\nu}^{2\nu} dy_1 \cdots \int_{t\nu}^{(t+1)\nu} dy_t \chi(x + y_1) \cdots \chi(x + y_t)
\]

\[
= \nu^{-t} \int_U dx I_\nu(x + \nu) I_\nu(x + 2\nu) \cdots I_\nu(x + t\nu)
\]

\[
= \nu^{-t} \int_U J_\nu(x) \, dx < \nu^{-t}(t\nu)^t = \epsilon^t.
\]

DEPARTMENT OF MATHEMATICS, UNIVERSITY OF COLORADO, BOULDER, COLORADO 80302