MAXIMAL ASYMPTOTIC NONBASES

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ABSTRACT. Let $A$ be a set of nonnegative integers. If all but a finite number of positive integers can be written as a sum of $h$ elements of $A$, then $A$ is an asymptotic basis of order $h$. Otherwise, $A$ is an asymptotic nonbasis of order $h$. A class of maximal asymptotic nonbases is constructed, and it is proved that any asymptotic nonbasis of order 2 that satisfies a certain finiteness condition is a subset of a maximal asymptotic nonbasis of order 2.

Let $A$ be a set of nonnegative integers containing 0. The $h$-fold sum of $A$, denoted $hA$, is the set of all sums of $h$ not necessarily distinct elements of $A$. If $hA$ contains all but a finite number of positive integers, then $A$ is an asymptotic basis of order $h$. The set $A$ is a minimal asymptotic basis of order $h$ if $A$ is an asymptotic basis of order $h$, but $A \setminus \{a\}$ is not an asymptotic basis of order $h$ for every $a \in A$. Examples of minimal asymptotic bases were constructed in [1], and also an example of an asymptotic basis which contains no subset that is a minimal asymptotic basis.

The set $A$ is an asymptotic nonbasis of order $h$ if $A$ is not an asymptotic basis of order $h$. If $A$ is an asymptotic nonbasis of order $h$, but $A \cup \{a\}$ is an asymptotic basis of order $h$ for every nonnegative integer $a \notin A$, then $A$ is a maximal asymptotic nonbasis of order $h$. Maximal asymptotic nonbases were constructed in [1] by taking finite unions of the nonnegative parts of congruence classes. In this paper we construct a new class of maximal asymptotic nonbases that are not unions of congruence classes, and we prove that every asymptotic nonbasis of order 2 that satisfies a certain finiteness condition is a subset of a maximal asymptotic nonbasis of order 2. We do not know whether every asymptotic nonbasis is a subset of a maximal asymptotic nonbasis, nor whether there exist maximal asymptotic nonbases with zero density.

Let $[a, b]$ denote the set of integers $n$ such that $a \leq n \leq b$. 

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Theorem 1. Let \( h \geq 2 \), and let \( n_1 < n_2 < \cdots \) be an increasing sequence of positive integers such that \( h^2 n_t + 2h \leq n_{t+1} \). Let

\[
A = [0, n_1] \cup \bigcup_{t=1}^{\infty} [bn_t + 2, n_{t+1}].
\]

Then there exists a maximal asymptotic nonbasis \( A^* \) of order \( h \) such that \( A \subseteq A^* \) and \( hA = hA^* \).

Proof. We shall construct an increasing sequence \( A = A_0 \subset A_1 \subset A_2 \subset \cdots \) of asymptotic nonbases of order \( h \) and two increasing sequences of positive integers \( m_1 < m_2 < \cdots \) and \( q_1 < q_2 < \cdots \) such that

(i) \( m_1 < m_2 < \cdots < m_k \) are the \( k \) smallest integers not in \( A_k \);
(ii) \( A_k \cup \{m_k\} \) is an asymptotic basis of order \( h \);
(iii) \( hA_k = hA_k \) for all \( k \) and
(iv) \( q_j \notin (h-1)A_k \) for all \( j \in [1, k] \).

Let \( A^* = \bigcup_{k=0}^{\infty} A_k \). Clearly, \( hA \subset hA^* \), since \( A = A_0 \subset A^* \). If \( n \in hA^* \), then \( n \in hA_k \) for some \( k \), and so \( n \in hA \) by (iii). Therefore, \( hA^* = hA \), and \( A^* \) is an asymptotic nonbasis of order \( h \). Let \( m \notin A^* \). Then \( m \leq m_k \) for some \( k \), and \( m \notin A_k \). It follows from (i) that \( m = m_j \) for some \( j \in [1, k] \), and from (ii) that \( A^* \cup \{m\} \) is an asymptotic basis of order \( h \). Therefore, \( A^* \) is a maximal asymptotic nonbasis of order \( h \) such that \( hA = hA^* \).

We construct the sequences \( \{A_k\} \), \( \{m_k\} \), and \( \{q_k\} \) inductively. Clearly, \( hA \) consists of all nonnegative integers except those of the form \( hn_t + 1 \). Let \( m_1 \) be the largest positive integer such that \( (h-1)(A \cup [0, m_1-1]) = (h-1)A \). Then \( (h-1)A \subset (h-1)(A \cup [0, m_1]) \). Let \( A'_1 = A \cup [0, m_1-1] \), and choose an integer \( q_1 \) in

\[
(b-1)(A'_1 \cup \{m_1\}) \setminus (b-1)A'_1 = (b-1)(A \cup [0, m_1]) \setminus (b-1)A.
\]

Let

\[
B_1 = \{bn_t + 1 - q_1 | bn_t + 1 - q_1 > \max(n_t, m_1, q_1)\}
\]

and let \( A_1 = A'_1 \cup B_1 \). Since \( [0, m_1-1] \subset A'_1 \subset A_1 \) and \( m_1 \notin B_1 \), it follows that \( m_1 \) is the smallest positive integer not in \( A_1 \). If \( hn_t + 1 \in hA_1 \), then \( hn_t + 1 \) is the sum of \( h \) elements of \( A_1 \), and at least one of these summands must be in the interval \([n_t + 1, hn_t + 1]\). But there is at most one element of \( A_1 \) in this interval, namely, \( hn_t + 1 - q_1 \), hence \( hn_t + 1 - q_1 \) must be one of the \( h \) summands of \( hn_t + 1 \). Then the sum of the \( b-1 \) remaining summands must be \( q_1 \). Since all elements of \( B_1 \) are greater than \( q_1 \), these summands are all elements of \( A'_1 \). But \( q_1 \notin (h-1)A'_1 \). Therefore, \( hn_t + 1 \notin hA_1 \), and so \( hA = hA_1 \). But
$$q_1 \in (b-1)(A'_1 \cup \{m_1\}) \subset (b-1)(A_1 \cup \{m_1\}),$$

and so $A_1 \cup \{m_1\}$ is an asymptotic basis of order $h$. Therefore, the integers $m_1$ and $q_1$ and the asymptotic nonbasis $A_1$ satisfy conditions (i)-(iv).

Now suppose that integers $m_1 < \cdots < m_{k-1}$ and $q_1 < \cdots < q_{k-1}$ and asymptotic nonbases $A = A_0 \subset A_1 \subset \cdots \subset A_{k-1}$ satisfy conditions (i)-(iv).

If $(h-1)(A_{k-1} \cup \{m_{k-1} + 1\}) \neq (h-1)A_{k-1}$, let $m_k = m_{k-1} + 1$. Otherwise, let $m_k$ be the largest integer such that $m_k > m_{k-1}$ and

$$(h-1)(A_{k-1} \cup [m_{k-1} + 1, m_k - 1]) = (h-1)A_{k-1}.$$

Let $A'_k = A_{k-1} \cup \{m_{k-1} + 1, m_k - 1\}$. Then $(h-1)A_{k-1} = (h-1)A'_k \subseteq (h-1)(A'_k \cup \{m_k\})$. Choose an integer $q_k$ in $(h-1)(A'_k \cup \{m_k\})\backslash(h-1)A'_k$, and let

$$B_k = \{hn_t + 1 - q_k | h, m_k, q_1, \ldots, q_k\}.$$ 

Now let $A_k = A'_k \cup B_k$. Since $A_k \setminus A_{k-1}$ consists of integers all greater than $m_{k-1}$, and since $[m_{k-1} + 1, m_k - 1] \subset A'_k \subset A_k$, it follows that $m_1 < \cdots < m_{k-1} < m_k$ are the $k$ smallest integers not in $A_k$. If $hn_t + 1 \in hA_k$, then $hn_t + 1$ is the sum of $h$ elements of $A_k$, at least one of which must be in the interval $[n_t + 1, hn_t + 1]$. But the only such elements of $A_k$ are of the form $hn_t + 1 - q_j$ for $j \in [1, k]$. Since the elements of $B_k$ are all larger than every $q_j$, it follows that $q_j \in (h-1)A'_k$ for some $j \in [1, k]$. But $q_k \notin (h-1)A'_k$, and, since $(h-1)A'_k = (h-1)A_{k-1}$, also $q_j \notin (h-1)A'_k$ for $j \in [1, k-1]$. Therefore, $hn_t + 1 \notin hA_k$, and so $hA_k = hA$. But $q_k \in (h-1)(A'_k \cup \{m_k\}) \subset (h-1)(A_k \cup \{m_k\})$, and so $A_k \cup \{m_k\}$ is an asymptotic basis of order $h$.

Thus, the integers $m_k$ and $q_k$ and the set $A_k$ satisfy conditions (i)-(iv). This completes the induction.

Remark. Since $A$ contains arbitrarily long sequences of consecutive integers, and $A \subset A^*$, the maximal asymptotic nonbasis $A^*$ is not a finite union of the nonnegative parts of congruence classes.

Theorem 2. Let $A$ be an asymptotic nonbasis of order $h$ such that $A \cup F$ is an asymptotic nonbasis of order $h$ for any finite set $F$ of nonnegative integers. Then $A \subset A^*$, where $A^*$ is an asymptotic nonbasis of order $h$ such that, for every integer $x \notin (h-1)A^*$, the set $A^* \cup \{x\}$ is an asymptotic basis of order $h$.

Proof. We shall construct a sequence $A = A_0 \subset A_1 \subset A_2 \subset \cdots$ of asymptotic nonbases of order $h$, and an increasing sequence of positive integers $n_1 < n_2 < \cdots$ such that

(i) $A_k \setminus A_{k-1}$ is a finite set of positive integers all larger than $n_{k-1}$;

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(ii) \( n_1 < n_2 < \cdots < n_k \) are the \( k \) smallest integers not in \( hA_k \); and

(iii) if \( 0 < x < n_k/2 \) and \( x \notin (h - 1)A_k \), then \( n_k - x \in A_k \).

Let \( A^* = \bigcup_{k=0}^{\infty} A_k \). By (i) and (ii), the set \( hA^* \) does not contain the numbers \( n_1, n_2, \ldots \), and so \( A^* \) is an asymptotic nonbasis of order \( h \). If \( x \notin (h - 1)A^* \), then \( x \notin (h - 1)A_k \) for all \( k \). Choose \( n_k > 2x \). Then \( n_k - x \in A_k \subset A^* \) by (iii), and so \( n_k \notin 2(A^* \cup \{x\}) \subset h(A^* \cup \{x\}) \), since \( 0 \in A \subset A^* \). Therefore, \( A^* \cup \{x\} \) is an asymptotic basis of order \( h \) for every positive integer \( x \notin (h - 1)A^* \).

We construct the sequences \( \{A_k\} \) and \( \{n_k\} \) inductively. Suppose that integers \( n_1 < \cdots < n_{k-1} \) and asymptotic nonbases \( A = A_0 \subset A_1 \subset \cdots \subset A_{k-1} \) satisfy conditions (i)–(iii). Let \( A_k = A_{k-1} \cup [n_{k-1} + 1, 2n_{k-1}] \). By (i), \( A_k \setminus A \) is finite, and so the set \( A_k \) is an asymptotic nonbasis of order \( h \). Let \( n_k \) be the smallest integer such that \( n_k > n_{k-1} \) and \( n_k \notin hA_k^* \). Then \( n_k > 2n_{k-1} \). Let \( F_k \) be a maximal subset of the interval \( [n_k/2, n_k] \) such that \( n_k \notin h(A_k^* \cup F_k) \). Let \( A_k = A_k^* \cup F_k \). Clearly, the set \( A_k \) satisfies conditions (i) and (ii). If \( 0 < x < n_k/2 \) and \( x \notin (h - 1)A_k \), then \( n_k - x \notin [n_k/2, n_k] \), and so \( F_k \cup \{n_k - x\} \subset [n_k/2, n_k] \) and \( n_k \notin h(A_k^* \cup F_k \cup \{n_k - x\}) \). It follows from the maximality of \( F_k \) that \( n_k - x \notin F_k \subset A_k \). Therefore, \( A_k \) satisfies condition (iii), and the induction is complete.

**Corollary.** Let \( A \) be an asymptotic nonbasis of order 2 such that \( A \cup F \) is an asymptotic nonbasis of order 2 for every finite set \( F \) of nonnegative integers. Then \( A \) is a subset of a maximal asymptotic nonbasis of order 2.

**Remark.** The Corollary suggests the following problem. If \( A \) is an asymptotic basis of order 2 such that \( A \setminus F \) is also an asymptotic basis of order 2 for every finite subset \( F \) of \( A \), then does \( A \) contain a subset that is a minimal asymptotic basis of order 2?

**REFERENCE**


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